Contents

Contents i

List of Tables ii

List of Figures iii

8 The Boltzmann Equation 1

8.1 References . . . 1

8.2 Equilibrium, Nonequilibrium and Local Equilibrium . . . 2

8.3 Boltzmann Transport Theory . . . 4

8.3.1 Derivation of the Boltzmann equation . . . 4

8.3.2 Collisionless Boltzmann equation . . . 6

8.3.3 Scattering processes . . . 7

8.3.4 Collisional invariants . . . 9

8.3.5 Detailed balance . . . 10

8.3.6 Kinematics and cross section . . . 12

8.3.7 $\mathcal{H}$-theorem . . . 13

8.4 Weakly Inhomogeneous Gas . . . 15

8.5 Relaxation Time Approximation . . . 18

8.5.1 Approximation of collision integral . . . 18

8.5.2 Computation of the scattering time . . . 18

8.5.3 Thermal conductivity . . . 19

8.5.4 Viscosity . . . 21
8.5.5 Oscillating external force . . . 23
8.5.6 Quick and dirty calculation of transport coefficients . . . 24
8.5.7 Thermal diffusivity, kinematic viscosity, and Prandtl number . . . 25
8.6 Diffusion and the Lorentz model . . . 26
8.6.1 Failure of the relaxation time approximation . . . 26
8.6.2 Modified Boltzmann equation and its solution . . . 27
8.7 Linearized Boltzmann Equation . . . 29
8.7.1 Linearizing the collision integral . . . 29
8.7.2 Linear algebraic properties of $\hat{L}$ . . . 30
8.7.3 Steady state solution to the linearized Boltzmann equation . . . 32
8.7.4 Variational approach . . . 32
8.8 The Equations of Hydrodynamics . . . 36
8.9 Nonequilibrium Quantum Transport . . . 37
8.9.1 Boltzmann equation for quantum systems . . . 37
8.9.2 The heat equation . . . 41
8.9.3 Calculation of Transport Coefficients . . . 42
8.9.4 Onsager Relations . . . 43
8.10 Appendix I: The BBGKY Hierarchy and the Boltzmann Equation . . . 44
8.10.1 BBGKY hierarchy . . . 45
8.10.2 Boltzmann equation . . . 46
8.11 Appendix II: Boltzmann Equation and Collisional Invariants . . . 50

List of Tables

8.1 Viscosities, thermal conductivities, and Prandtl numbers for some common gases 26

List of Figures

8.1 Level sets for a sample $f(\bar{x}, \bar{p}, \bar{t}) = C\, e^{-\frac{1}{2}(\bar{x}-\bar{p}\bar{t})^2}\, e^{-\frac{1}{2}\bar{p}^2}$ 7
8.2 One and two particle scattering processes 9
8.3 Graphic representation of the equation $n\,\sigma\,\bar{v}_{\mathrm{rel}}\,\tau = 1$ 19
8.4 *Gedankenexperiment* to measure shear viscosity η in a fluid 22
8.5 Experimental data on thermal conductivity and shear viscosity 23
8.6 Scattering in the center of mass frame 34
8.7 Thermocouple and Peltier effect 40

Chapter 8

The Boltzmann Equation

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8.2 Equilibrium, Nonequilibrium and Local Equilibrium

Classical equilibrium statistical mechanics is described by the full N-body distribution,

$$\varrho_N(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N;\boldsymbol{p}_1,\ldots,\boldsymbol{p}_N) = \frac{1}{N!}\times\begin{cases} Z_N^{-1}\,e^{-\beta\hat{H}_N(\{\boldsymbol{p}_i\},\{\boldsymbol{x}_i\})} & \text{OCE} \\ \Xi^{-1}\,e^{\beta\mu N}\,e^{-\beta\hat{H}_N(\{\boldsymbol{p}_i\},\{\boldsymbol{x}_i\})} & \text{GCE} \end{cases}, \tag{8.1}$$

We assume a Hamiltonian of the form

$$\hat{H}_N = \sum_{i=1}^N \frac{\boldsymbol{p}_i^2}{2m} + \sum_{i=1}^N v(\boldsymbol{x}_i) + \sum_{i<j}^N u(\boldsymbol{x}_i-\boldsymbol{x}_j). \tag{8.2}$$

Here $v(\boldsymbol{x}) = U_{\rm ext}(\boldsymbol{x})$ is due to external forces. In the context of transport theory, $v(\boldsymbol{x})$ typically will denote the effect of an applied external field, *e.g.* $v(\boldsymbol{x}) = -q\boldsymbol{E}\cdot\boldsymbol{x}$ for a particle of charge q in the presence of a uniform electric field $\boldsymbol{E}$. We write $\boldsymbol{x}_i$ and $\boldsymbol{p}_i$ for the corresponding phase space variables, the position and momentum vectors for the $i^{\rm th}$ particle, respectively. The quantity

$$\varrho_N(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N,\boldsymbol{p}_1,\ldots,\boldsymbol{p}_N)\prod_{j=1}^N d\mu_j \quad , \tag{8.3}$$

with $d\mu_j \equiv d^dx_j\,d^dp_j/h^d$, is the probability, under equilibrium conditions, of finding N particles in the system, with particle #1 lying within d^dx_1 of $\boldsymbol{x}_1$ and having momentum within d^dp_1 of $\boldsymbol{p}_1$, *etc.* The temperature T and chemical potential μ are constants, independent of position. Note that $\varrho_N(\{\boldsymbol{x}_i\},\{\boldsymbol{p}_i\})$ is dimensionless.

Nonequilibrium statistical mechanics seeks to describe thermodynamic systems which are out of equilibrium, meaning that the distribution function is not given by the Boltzmann distribution above. Rather, it is a time-dependent quantity, ϱ_N. In a general nonequilibrium setting, it is hopeless to make progress – we'd have to integrate the equations of motion for all the constituent particles. However, typically we are concerned with situations where external forces or constraints are imposed over some macroscopic scale. Examples would include the imposition of a voltage drop across a metal, or a temperature differential across any thermodynamic sample. In such cases, scattering at microscopic length and time scales described by the *mean free path* ℓ and the *collision time* τ work to establish *local equilibrium* throughout the system. A local equilibrium is a state described by a space and time varying temperature $T(\boldsymbol{r},t)$, chemical potential $\mu(\boldsymbol{r},t)$, and flow velocity $\boldsymbol{V}(\boldsymbol{r},t)$. As we will see, the Boltzmann distribution with spatiotemporally varying T, μ, and $\boldsymbol{V}$ will in general not be a solution to the evolution equation governing the distribution function. Rather, the distribution for systems slightly out of equilibrium will be of the form $f = f^0 + \delta f$, where f^0 describes a state of local equilibrium.

We will mainly be interested in the one-body distribution

$$\begin{aligned} f(\mathbf{x},\mathbf{p};t) &= \sum_{i=1}^N \big\langle\,\delta\big(\mathbf{x}-\boldsymbol{x}_i(t)\big)\,\delta\big(\mathbf{p}-\boldsymbol{p}_i(t)\big)\,\big\rangle \\ &= Nh^{-d}\int\prod_{i=2}^N d\mu_i\;\varrho_N(\mathbf{x},\boldsymbol{x}_2,\ldots,\boldsymbol{x}_N;\mathbf{p},\boldsymbol{p}_2,\ldots,\boldsymbol{p}_N,t) \quad . \end{aligned} \tag{8.4}$$

Note that $f(\mathbf{x},\mathbf{p},t)$ has dimensions of h^{-d}, and $f(\mathbf{x},\mathbf{p},t)\,d^d\mathrm{x}\,d^d\mathrm{p}$ is the average number of particles found within $d^d\mathrm{x}$ of $\mathbf{x}$ and $d^d\mathrm{p}$ of $\mathbf{p}$ at time t. The full k-particle distributions, are given by

$$\begin{aligned} f_k\big(\{\mathbf{x}_i\},\{\mathbf{p}_i\},t\big) &= {\sum_{\{j_1,\ldots,j_k\}}}' \big\langle \delta\big(\mathbf{x}_1 - \boldsymbol{x}_{j_1}(t)\big)\cdots\delta\big(\mathbf{x}_k - \boldsymbol{x}_{j_k}(t)\big)\,\delta\big(\mathbf{p}_1 - \boldsymbol{p}_{j_1}(t)\big)\cdots\delta\big(\mathbf{p}_k - \boldsymbol{p}_{j_k}(t)\big)\big\rangle \\ &= \frac{N!\,h^{-kd}}{(N-k)!}\int\prod_{i=k+1}^{N} d\mu_i\;\varrho_N\big(\mathbf{x}_1,\ldots,\mathbf{x}_k,\boldsymbol{x}_{k+1},\ldots,\boldsymbol{x}_N;\mathbf{p}_1,\ldots,\mathbf{p}_k,\boldsymbol{p}_{k+1},\ldots,\boldsymbol{p}_N\,,\,t\big) \quad , \end{aligned} \tag{8.5}$$

where $\{\mathbf{x}_j\}$ as an argument of the k-particle distribution f_k denotes the ordered set $\{\mathbf{x}_1,\ldots,\mathbf{x}_k\}$, where $k\in\{1,\ldots,N\}$ (similarly for $\{\mathbf{p}_j\}$). The prime on the sum over the indices $\{j_1,\ldots,j_k\}$ indicates that no two indices take the same value. Note that the normalization of f_k is

$$\int\prod_{i=1}^{k} d^d\mathrm{x}_i\,d^d\mathrm{p}_i\;f_k\big(\{\mathbf{x}_j\},\{\mathbf{p}_j\},t\big) = \frac{N!}{(N-k)!} \quad , \tag{8.6}$$

for all t. We write the one-body distribution $f_1(\mathbf{x}_1,\mathbf{p}_1,t)\equiv f(\mathbf{x},\mathbf{p},t)$, where $\mathbf{x}=\mathbf{x}_1$ and $\mathbf{p}=\mathbf{p}_1$. Henceforth we will drop the distinction between boldface Roman symbols $\{\mathbf{x},\mathbf{p}\}$ and boldface math italic $\{\boldsymbol{x},\boldsymbol{p}\}$, as it will no longer be needed.

In the GCE, we sum over different particle numbers N. Assuming $v=0$ so that there is no one-body potential to break translational symmetry, the equilibrium distribution is time-independent and space-independent:

$$f^0(\boldsymbol{r},\boldsymbol{p}) = n\,(2\pi m k_{\mathrm{B}}T)^{-d/2}\,e^{-\boldsymbol{p}^2/2mk_{\mathrm{B}}T} = h^{-d}\,z\,e^{-\boldsymbol{p}^2/2mk_{\mathrm{B}}T} \quad , \tag{8.7}$$

where $n=N/V$ or $n=n(T,\mu)$ is the particle density in the OCE or GCE, with $z=\exp(\mu/k_{\mathrm{B}}T)=n\lambda_T^d$ the fugacity for ideal classical point particles. From the one-body distribution we can compute things like the particle current density $\boldsymbol{j}$ and the energy current density $\boldsymbol{j}_\varepsilon$:

$$\boldsymbol{j}(\boldsymbol{r},t) = \int d^dp\;f(\boldsymbol{r},\boldsymbol{p};t)\,\frac{\boldsymbol{p}}{m} \tag{8.8}$$

$$\boldsymbol{j}_\varepsilon(\boldsymbol{r},t) = \int d^dp\;f(\boldsymbol{r},\boldsymbol{p};t)\;\varepsilon(\boldsymbol{p})\,\frac{\boldsymbol{p}}{m} \quad , \tag{8.9}$$

where $\varepsilon(\boldsymbol{p})=\boldsymbol{p}^2/2m$. Both quantities vanish in equilibrium, when $f=f^0$, since $f^0(\boldsymbol{r},\boldsymbol{p})$ depends only on $\boldsymbol{p}^2$ and not on the direction of $\boldsymbol{p}$. In a *steady state* nonequilibrium situation, the above quantities are time-independent.

Thermodynamics says that

$$T\,dS = dE + p\,dV - \mu\,dN \quad , \tag{8.10}$$

which may be cast as one among the corresponding current densities. Suppressing the explicit $\boldsymbol{r}$ and t dependence,

$$T\,\boldsymbol{j}_s = \boldsymbol{j}_\varepsilon - \mu\,\boldsymbol{j} \quad , \tag{8.11}$$

where $\boldsymbol{j}_s$ is the entropy current density. Some authors define the *conductive heat current density* $\boldsymbol{j}_q$ (sometimes written as $\boldsymbol{q}$) as $\boldsymbol{j}_q=\boldsymbol{j}_\varepsilon - h\boldsymbol{j}$, where h is the enthalpy per particle. Since $h=\mu+Ts$, where s is the

entropy per particle, we have the relation $\boldsymbol{j}_q = T\boldsymbol{j}_s - Ts\boldsymbol{j}$. To make matters even more confusing, some sources (particularly in condensed matter physics) define a "thermodynamic heat current density" as $\boldsymbol{j}_{\rm th} \equiv \boldsymbol{j}_\varepsilon - \mu\,\boldsymbol{j} = T\boldsymbol{j}_s$. When there is no particle current and $\boldsymbol{j} = 0$, we have $\boldsymbol{j}_\varepsilon = \boldsymbol{j}_q = \boldsymbol{j}_{\rm th} = T\boldsymbol{j}_s$.

When the individual particles are not point particles, they possess angular momentum as well as linear momentum. Following Lifshitz and Pitaevskii, we abbreviate $\Gamma = (\boldsymbol{p}, \boldsymbol{L})$ for these two variables for the case of diatomic molecules, and $\Gamma = (\boldsymbol{p}, \boldsymbol{L}, \hat{\boldsymbol{n}} \cdot \boldsymbol{L})$ in the case of symmetric top molecules, where $\hat{\boldsymbol{n}}$ is the symmetry axis of the top. We then have, in $d = 3$ dimensions,

$$d\Gamma = \begin{cases} d^3p & \text{point particles} \\ d^3p\, L\, dL\, d\Omega_{\hat{\boldsymbol{L}}} & \text{diatomic molecules} \\ d^3p\, L^2\, dL\, d\Omega_{\hat{\boldsymbol{L}}}\, d\cos\vartheta & \text{symmetric tops} \quad , \end{cases} \tag{8.12}$$

where $\vartheta = \cos^{-1}(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{L}})$. We will call the set Γ the 'kinematic variables'. The instantaneous number density at $\boldsymbol{r}$ is then

$$n(\boldsymbol{r}, t) = \int d\Gamma\, f(\boldsymbol{r}, \Gamma; t) \quad . \tag{8.13}$$

One might ask why we do not also keep track of the angular orientation of the individual molecules. There are two reasons. First, the rotations of the molecules are generally extremely rapid, so we are justified in averaging over these motions. Second, the orientation of, say, a rotor does not enter into its energy. While the same can be said of the spatial position in the absence of external fields, (i) in the presence of external fields one must keep track of the position coordinate $\boldsymbol{r}$ since there is physical transport of particles from one region of space to another, and (ii) the collision process, which as we shall see enters the dynamics of the distribution function, takes place in real space.

8.3 Boltzmann Transport Theory

8.3.1 Derivation of the Boltzmann equation

For simplicity of presentation, we assume point particles in three space dimensions. Recall that

$$f(\boldsymbol{r}, \boldsymbol{p}, t)\, d^3r\, d^3p \equiv \begin{cases} \text{\# of particles with positions within } d^3r \text{ of} \\ \boldsymbol{r} \text{ and momenta within } d^3p \text{ of } \boldsymbol{p} \text{ at time } t. \end{cases} \tag{8.14}$$

Thus, the units of $f(\boldsymbol{r}, \boldsymbol{p}, t)$ are those of inverse action cubed, *i.e.* $(ML^2/T)^{-3}$. We now ask how the distribution function $f(\boldsymbol{r}, \boldsymbol{p}, t)$ evolves in time. It is clear that in the absence of collisions, the distribution function must satisfy the continuity equation,

$$\frac{\partial f}{\partial t} + \boldsymbol{\nabla} \cdot (\boldsymbol{u} f) = 0 \quad . \tag{8.15}$$

This is just the condition of number conservation for particles. Take care to note that $\boldsymbol{\nabla}$ and $\boldsymbol{u}$ are *six*-dimensional *phase space* vectors:

$$\begin{aligned} \boldsymbol{u} &= (\ \dot{x} \quad , \ \dot{y} \quad , \ \dot{z} \quad , \ \dot{p}_x \quad , \ \dot{p}_y \quad , \ \dot{p}_z\ ) \\ \boldsymbol{\nabla} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_z} \right) \quad . \end{aligned} \tag{8.16}$$

The continuity equation describes a distribution in which each constituent particle evolves according to a prescribed dynamics, which for a mechanical system is specified by

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial H}{\partial \boldsymbol{p}} = \boldsymbol{v}(\boldsymbol{p}) \qquad , \qquad \frac{d\boldsymbol{p}}{dt} = -\frac{\partial H}{\partial \boldsymbol{r}} = \boldsymbol{F} \quad , \tag{8.17}$$

where $\boldsymbol{F}$ is an external applied force. Here,

$$H(\boldsymbol{p}, \boldsymbol{r}) = \varepsilon(\boldsymbol{p}) + U(\boldsymbol{r}) \quad . \tag{8.18}$$

For example, under the influence of gravity, $U(\boldsymbol{r}) = m\boldsymbol{g} \cdot \boldsymbol{r}$ and $\boldsymbol{F} = -\boldsymbol{\nabla} U = -m\boldsymbol{g}$.

Note that as a consequence of the dynamics, we have $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, *i.e.* phase space flow is *incompressible*[1]. Thus, in the absence of collisions, we have

$$\frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} f = 0 \quad . \tag{8.19}$$

The differential operator $D_t \equiv \partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla}$ is sometimes called the 'convective derivative', because $D_t f$ is the time derivative of f in a comoving frame of reference. To see this, evaluate the time derivative of the time-dependent distribution $f(\boldsymbol{r}, \boldsymbol{p}, t)$ along a particle trajectory, *i.e.* substituting $\boldsymbol{r} \to \boldsymbol{r}(t)$ and $\boldsymbol{p} = \boldsymbol{p}(t)$. We have

$$\frac{d}{dt} f\big(\boldsymbol{r}(t), \boldsymbol{p}(t), t\big) = \frac{\partial f}{\partial \boldsymbol{r}} \cdot \frac{d\boldsymbol{r}}{dt}\bigg|_{\{\boldsymbol{r}(t), \boldsymbol{p}(t), t\}} + \frac{\partial f}{\partial \boldsymbol{p}} \cdot \frac{d\boldsymbol{p}}{dt}\bigg|_{\{\boldsymbol{r}(t), \boldsymbol{p}(t), t\}} + \frac{\partial f}{\partial t}\bigg|_{\{\boldsymbol{r}(t), \boldsymbol{p}(t), t\}} = \frac{Df}{Dt}\bigg|_{\{\boldsymbol{r}(t), \boldsymbol{p}(t), t\}} \quad . \tag{8.20}$$

Next we must consider the effect of collisions, which are not accounted for by the semiclassical dynamics. In a collision process, a particle with momentum $\boldsymbol{p}$ and one with momentum $\tilde{\boldsymbol{p}}$ can instantaneously convert into a pair with momenta $\boldsymbol{p}'$ and $\tilde{\boldsymbol{p}}'$, provided total momentum is conserved: $\boldsymbol{p} + \tilde{\boldsymbol{p}} = \boldsymbol{p}' + \tilde{\boldsymbol{p}}'$. This means that $D_t f \neq 0$. Rather, we should write

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \dot{\boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \dot{\boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = \left(\frac{df}{dt}\right)_{\rm coll} \tag{8.21}$$

where the right side is known as the *collision integral*. The collision integral is in general a *function* of $\boldsymbol{r}$, $\boldsymbol{p}$, and t and a *functional* of the distribution f. *Thus, in the absence of collisions, the convective derivative of the distribution $f(\boldsymbol{r}, \boldsymbol{p}, t)$ vanishes, meaning that the one-body distribution does not vary in time along a particle trajectory.*

We can write the Boltzmann equation as

$$\frac{\partial f}{\partial t} + \left(\frac{df}{dt}\right)_{\rm str} = \left(\frac{df}{dt}\right)_{\rm coll} \quad , \tag{8.22}$$

where

$$\left(\frac{df}{dt}\right)_{\rm str} \equiv \dot{\boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \dot{\boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} \tag{8.23}$$

is known as the *streaming term*. Thus, there are two contributions to $\partial f / \partial t$: streaming and collisions.

[1] This remains true even if $\varepsilon(\boldsymbol{p}, \boldsymbol{r})$ is a function of both momentum and position.

8.3.2 Collisionless Boltzmann equation

In the absence of collisions, the Boltzmann equation is given by

$$\frac{\partial f}{\partial t} + \frac{\partial \varepsilon}{\partial \boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \boldsymbol{\nabla} U \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0 \quad . \tag{8.24}$$

In order to gain some intuition about how the streaming term affects the evolution of the distribution $f(\boldsymbol{r}, \boldsymbol{p}, t)$, consider a case where $\boldsymbol{F} = 0$. We then have

$$\frac{\partial f}{\partial t} + \boldsymbol{v}(\boldsymbol{p}) \cdot \frac{\partial f}{\partial \boldsymbol{r}} = 0 \quad . \tag{8.25}$$

Clearly, then, *any* function of the form

$$f(\boldsymbol{r}, \boldsymbol{p}, t) = \varphi\big(\boldsymbol{r} - \boldsymbol{v}(\boldsymbol{p})\, t\,,\, \boldsymbol{p}\big) \tag{8.26}$$

will be a solution to the collisionless Boltzmann equation, where $\boldsymbol{v}(\boldsymbol{p}) = \partial\varepsilon/\partial\boldsymbol{p}$. One possible solution would be the Boltzmann distribution,

$$f(\boldsymbol{r}, \boldsymbol{p}, t) = C\, e^{\mu/k_{\rm B}T} e^{-\varepsilon(\boldsymbol{p})/k_{\rm B}T} \quad , \tag{8.27}$$

which is time-independent[2]. Here, C is a constant with units of $(T/ML^2)^d$. For a ballistic dispersion, $\varepsilon(\boldsymbol{p}) = \boldsymbol{p}^2/2m$ and $C = h^{-3}$, in which case $\int d^dr\, d^dp\, f(\boldsymbol{r}, \boldsymbol{p}) = V\lambda_T^{-3} \exp(\mu/k_{\rm B}T) = \langle \hat{N} \rangle$.

For a slightly less trivial example, let the initial distribution be $\varphi(\boldsymbol{r}, \boldsymbol{p}) = C\, e^{-\boldsymbol{r}^2/2\sigma^2} e^{-\boldsymbol{p}^2/2\kappa^2}$, so that

$$f(\boldsymbol{r}, \boldsymbol{p}, t) = C\, e^{-(\boldsymbol{r} - \frac{\boldsymbol{p}t}{m})^2/2\sigma^2}\, e^{-\boldsymbol{p}^2/2\kappa^2} \quad . \tag{8.28}$$

Consider the one-dimensional version, and rescale position, momentum, and time so that

$$f(x, p, t) = C\, e^{-(\bar{x} - \bar{p}\,\bar{t})^2/2}\, e^{-\bar{p}^2/2} \quad . \tag{8.29}$$

Now consider the level sets of f, where $f(x, p, t) = C\, e^{-\alpha^2/2}$. The equation for these sets is

$$\bar{x} = \bar{p}\,\bar{t} \pm \sqrt{\alpha^2 - \bar{p}^2} \quad . \tag{8.30}$$

For fixed $\bar{t}$, these level sets describe the loci in phase space of equal probability densities, with the probability density decreasing exponentially in the parameter α^2. For $\bar{t} = 0$, the initial distribution describes a Gaussian cloud of particles with a Gaussian momentum distribution. As $\bar{t}$ increases, the distribution widens in $\bar{x}$ but not in $\bar{p}$ – each particle moves with a constant momentum, so the set of momentum values never changes. However, the level sets in the $(\bar{x}\,,\,\bar{p})$ plane become elliptical, with a semimajor axis oriented at an angle $\theta = \frac{1}{2}\tan^{-1}(2/\bar{t})$ with respect to the $\bar{x}$ axis. For $\bar{t} > 0$, the particles at the outer edges of the cloud are more likely to be moving away from the center. See fig. 8.1.

Suppose we add in a constant external force $\boldsymbol{F}$. Then it is easy to show (and left as an exercise to the reader to prove) that any function of the form

$$f(\boldsymbol{r}, \boldsymbol{p}, t) = \varphi\left(\boldsymbol{r} - \frac{\boldsymbol{p}\,t}{m} + \frac{\boldsymbol{F}t^2}{2m}\,,\, \boldsymbol{p} - \boldsymbol{F}t\right) \tag{8.31}$$

satisfies the collisionless Boltzmann equation (ballistic dispersion assumed).

[2]Indeed, any arbitrary function of $\boldsymbol{p}$ alone would be a solution. Ultimately, we require some energy exchanging processes, such as collisions, in order for any initial nonequilibrium distribution to converge to the Boltzmann distribution.

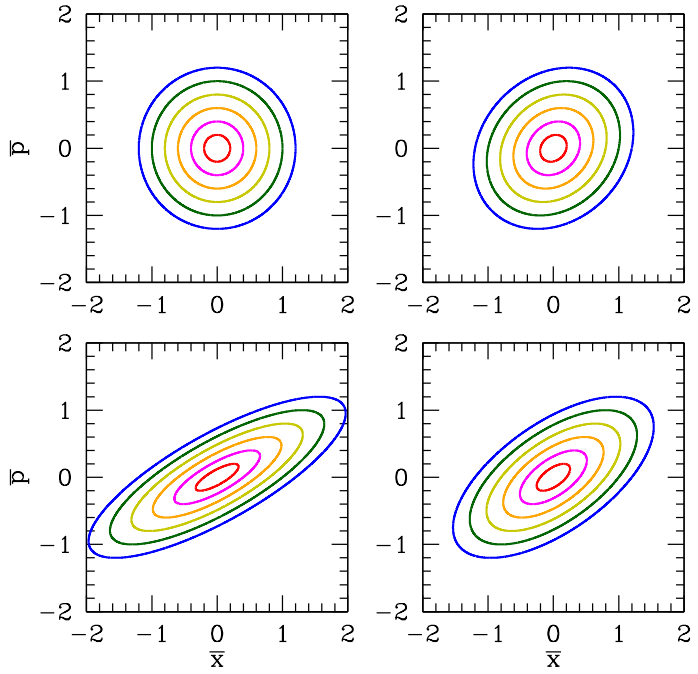


Figure 8.1: Level sets for a sample $f(\bar{x},\bar{p},\bar{t}) = C\,e^{-(\bar{x}-\bar{p}\bar{t})^2/2}e^{-\bar{p}^2/2}$, for values $f = C\,e^{-\alpha^2/2}$ with α in equally spaced intervals from $\alpha = 0.2$ (red) to $\alpha = 1.2$ (blue). The time variable $\bar{t}$ is taken to be $\bar{t} = 0.0$ (upper left), 0.2 (upper right), 0.8 (lower right), and 1.3 (lower left).

8.3.3 Scattering processes

What sort of processes contribute to the collision integral? There are two broad classes to consider. The first involves potential scattering, where a particle in state $|\Gamma\rangle$ scatters, in the presence of an external potential, to a state $|\Gamma'\rangle$. Recall that Γ is an abbreviation for the set of kinematic variables, *e.g.* $\Gamma = (\boldsymbol{p}, \boldsymbol{L})$ in the case of a diatomic molecule. For point particles, $\Gamma = (p_x, p_y, p_z)$ and $d\Gamma = d^3p$.

Single particle scattering

We now define the function $w\big(\Gamma' \,|\, \Gamma\big)$ such that

$$w\big(\Gamma' \,|\, \Gamma\big)\, f(\boldsymbol{r}, \Gamma; t)\, d\Gamma\, d\Gamma' = \text{rate per unit volume to scatter } \big\{|\,\Gamma + d\Gamma\,\rangle\big\} \to \big\{|\,\Gamma' + d\Gamma'\,\rangle\big\} \tag{8.32}$$

at time t. Here we have suppressed the time dependence in the distribution $f(\Gamma, t)$, and typically we will presume $w\big(\Gamma' \,|\, \Gamma\big)$ to be independent of both $\boldsymbol{r}$ and t. By $\big\{|\,\Gamma + d\Gamma\,\rangle\big\}$ we mean states with momenta within d^3p of $\boldsymbol{p}$ (more generally, within $d\Gamma = d^3p\, d^3L$ of $(\boldsymbol{p}, \boldsymbol{L})$) and at the same position coordinate $\boldsymbol{r}$. We assume the rate is independent of the position $\boldsymbol{r}$ and the time t. The units of $w\, d\Gamma$ are therefore $1/T$. The *differential scattering cross section* for single particle scattering is then

$$d\sigma = \frac{w\big(\Gamma' \,|\, \Gamma\big)}{n_{\rm s}\,|\boldsymbol{v}|}\, d\Gamma' \quad , \tag{8.33}$$

where $\boldsymbol{v} = \boldsymbol{p}/m$ is the particle's velocity and $n_{\rm s}$ the number density of scatterers.

In computing the collision integral for the state $|\boldsymbol{r}, \Gamma\rangle$, we must take care to sum over contributions from transitions *out of* this state, *i.e.* $|\Gamma\rangle \to |\Gamma'\rangle$, which reduce $f(\boldsymbol{r}, \Gamma)$, and transitions *into* this state, *i.e.* $|\Gamma'\rangle \to |\Gamma\rangle$, which increase $f(\boldsymbol{r}, \Gamma)$. Thus, for one-body scattering, we have

$$\frac{D}{Dt} f(\boldsymbol{r}, \Gamma; t) = \int d\Gamma' \left\{ w(\Gamma \,|\, \Gamma')\, f(\boldsymbol{r}, \Gamma'; t) - w(\Gamma' \,|\, \Gamma)\, f(\boldsymbol{r}, \Gamma; t) \right\} = \left(\frac{df}{dt} \right)_{\rm coll} \quad . \tag{8.34}$$

Mathematically, in the case of point particles, where $\Gamma = \boldsymbol{p}$, this is a linear, first-order integro-differential equation in seven variables $\{x, y, z, p_x, p_y, p_z, t\}$. It can be expressed as $\partial_t f = \mathcal{L} f$, where $\mathcal{L}$ is a linear integro-differential operator. With initial conditions $f(\boldsymbol{r}, \boldsymbol{p}, t = 0)$ this can in principle be integrated forward in time, and in the autonomous case where all the rates and external forces are time-independent, we have $f(t) = \exp(\mathcal{L}t)\, f(0)$.

Two particle scattering

The second class is that of two-particle scattering processes, *i.e.* $|\,\{\boldsymbol{r}, \Gamma\}, \{\boldsymbol{r}, \Gamma_1\}\,\rangle \to |\,\{\boldsymbol{r}, \Gamma'\}, \{\boldsymbol{r}, \Gamma_1'\}\,\rangle$. We define the scattering function $w\big(\Gamma', \Gamma_1' \,\big|\, \Gamma, \Gamma_1\big)$ by

$$w\big(\Gamma', \Gamma_1' \,\big|\, \Gamma, \Gamma_1\big)\, f_2(\boldsymbol{r}, \Gamma; \boldsymbol{r}, \Gamma_1; t)\, d\Gamma\, d\Gamma_1\, d\Gamma'\, d\Gamma_1' = \begin{cases} \text{rate per unit volume to scatter two particles} \\ \big\{|\,\Gamma + d\Gamma, \Gamma_1 + d\Gamma_1\,\rangle\big\} \to \big\{|\,\Gamma' + d\Gamma', \Gamma_1' + d\Gamma_1'\,\rangle\big\} \end{cases} \tag{8.35}$$

at time t. We assume that $w\big(\Gamma', \Gamma_1' \,\big|\, \Gamma, \Gamma_1\big)$ is independent of $\boldsymbol{r}$ and t. Thus the units of $w\, d\Gamma\, d\Gamma_1$ are L^d/T, and the differential scattering cross section is

$$d\sigma = \frac{w\big(\Gamma', \Gamma_1' \,\big|\, \Gamma, \Gamma_1\big)}{|\boldsymbol{v} - \boldsymbol{v}_1|}\, d\Gamma'\, d\Gamma_1' \quad . \tag{8.36}$$

For two-body scattering, we therefore have

$$\begin{aligned} \frac{D}{Dt} f(\boldsymbol{r}, \Gamma; t) = \int d\Gamma_1 \int d\Gamma' \int d\Gamma_1' \Big\{ & w\big(\Gamma, \Gamma_1 \,\big|\, \Gamma', \Gamma_1'\big)\, f_2(\boldsymbol{r}, \Gamma'; \boldsymbol{r}, \Gamma_1'; t) \\ & - w\big(\Gamma', \Gamma_1' \,\big|\, \Gamma, \Gamma_1\big)\, f_2(\boldsymbol{r}, \Gamma; \boldsymbol{r}, \Gamma_1; t) \Big\} = \left(\frac{df}{dt} \right)_{\rm coll} \quad . \end{aligned} \tag{8.37}$$

Unlike the one-body scattering case, the kinetic equation for two-body scattering does not close, since the LHS involves the one-body distribution $f \equiv f_1$ and the RHS involves the two-body distribution f_2. One can then write an equation for f_2 which also does not close – it involves f_3 – and so on. The result is known as the *BBGKY hierarchy*, and is discussed in some detail in the appendix §8.10.

In practice, an approximation is made in order to close the equations. The simplest and most common such approximation, known as the *Stosszahlansatz*, or *molecular chaos assumption*, is due to Boltzmann himself. We replace the two-body distribution by a product of one-body distributions, *viz.*[3]

$$f_2(\boldsymbol{r}, \Gamma; \boldsymbol{r}_1, \Gamma_1; t) \approx f(\boldsymbol{r}, \Gamma; t)\, f(\boldsymbol{r}_1, \Gamma_1; t) \quad . \tag{8.38}$$

[3] As discussed in the appendix §8.10, factorization is imposed on the *incoming* two-particle distribution immediately prior to a collision. Outgoing particles generally acquire correlations.

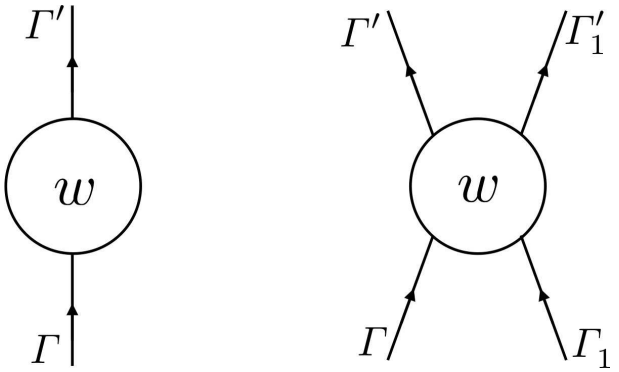


Figure 8.2: Left: single particle scattering process $|\Gamma\rangle \to |\Gamma'\rangle$. Right: two-particle scattering process $|\Gamma, \Gamma_1\rangle \to |\Gamma', \Gamma_1'\rangle$.

We then have

$$\frac{D}{Dt} f(\boldsymbol{r}, \Gamma; t) = \int d\Gamma_1 \int d\Gamma' \int d\Gamma_1' \Big\{ w\big(\Gamma, \Gamma_1 \,\big|\, \Gamma', \Gamma_1'\big)\, f(\boldsymbol{r}, \Gamma'; t)\, f(\boldsymbol{r}, \Gamma_1'; t) - w\big(\Gamma', \Gamma_1' \,\big|\, \Gamma, \Gamma_1\big)\, f(\boldsymbol{r}, \Gamma; t)\, f(\boldsymbol{r}, \Gamma_1; t) \Big\} = \left(\frac{df}{dt}\right)_{\rm coll} \quad . \tag{8.39}$$

Now we have a closed but *nonlinear* first-order integro-PDE. The nonlinearity adds a new wrinkle, and as we shall see below a further approximation involves *linearizing* the collision integral.

We stress that in both cases (*i.e.* single-particle and two-particle scattering) we assume that all scattering occurs *locally*, *i.e.* the particles attain their asymptotic kinematic states on distance scales small compared to the mean free path. In this case we can treat each scattering process independently. This assumption is particular to rarefied systems, *i.e.* gases, and is not appropriate for dense liquids. The two types of scattering processes are depicted in fig. 8.2.

Nota bene: All $w(\Gamma' \,|\, \Gamma)$ and all $w\big(\Gamma', \Gamma_1' \,\big|\, \Gamma, \Gamma_1\big)$ are non-negative!

8.3.4 Collisional invariants

Consider a function $A(\boldsymbol{r}, \boldsymbol{p})$ of position and momentum. Its average value at time t is

$$\langle A(t) \rangle = \int d^3r\, d^3p\, A(\boldsymbol{r}, \boldsymbol{p})\, f(\boldsymbol{r}, \boldsymbol{p}, t) \quad . \tag{8.40}$$

Taking the time derivative,

$$\begin{aligned} \frac{d\langle A(t)\rangle}{dt} &= \int d^3r \int d^3p\, A(\boldsymbol{r}, \boldsymbol{p})\, \frac{\partial f}{\partial t} \\ &= \int d^3r \int d^3p\, A(\boldsymbol{r}, \boldsymbol{p}) \left\{ -\frac{\partial}{\partial \boldsymbol{r}} \cdot (\dot{\boldsymbol{r}} f) - \frac{\partial}{\partial \boldsymbol{p}} \cdot (\dot{\boldsymbol{p}} f) + \left(\frac{df}{dt}\right)_{\rm coll} \right\} \\ &= \int d^3r \int d^3p \left\{ \left(\frac{\partial A}{\partial \boldsymbol{r}} \cdot \frac{d\boldsymbol{r}}{dt} + \frac{\partial A}{\partial \boldsymbol{p}} \cdot \frac{d\boldsymbol{p}}{dt} \right) f + A(\boldsymbol{r}, \boldsymbol{p}) \left(\frac{df}{dt}\right)_{\rm coll} \right\} \quad . \end{aligned} \tag{8.41}$$

Hence, if A is preserved by the dynamics between collisions, then[4]

$$\frac{dA(\boldsymbol{r}(t),\boldsymbol{p}(t))}{dt} = \frac{\partial A}{\partial \boldsymbol{r}}\cdot\frac{d\boldsymbol{r}}{dt} + \frac{\partial A}{\partial \boldsymbol{p}}\cdot\frac{d\boldsymbol{p}}{dt} = 0 \quad . \tag{8.42}$$

We therefore have that the rate of change of $\langle A(t)\rangle$ is determined wholly by the collision integral

$$\frac{d\langle A(t)\rangle}{dt} = \int d^3r \int d^3p\, A(\boldsymbol{r},\boldsymbol{p}) \left(\frac{df}{dt}\right)_{\rm coll} \quad . \tag{8.43}$$

Conservation of A by the collisions themselves means

$$A(\boldsymbol{p}) + A(\boldsymbol{p}_1) = A(\boldsymbol{p}') + A(\boldsymbol{p}_1') \quad , \tag{8.44}$$

for the process $(\boldsymbol{p},\boldsymbol{p}_1) \to (\boldsymbol{p}',\boldsymbol{p}_1')$. A *collisional invariant* is a one-particle function whose sum over the colliding particles is unchanged by every allowed collision. Examples of collisional invariants include the particle number ($A = 1$), the components of the total momentum ($A = p_\mu$) (in the absence of broken translational invariance, due *e.g.* to the presence of walls), and the total energy ($A = \varepsilon(\boldsymbol{p})$).

8.3.5 Detailed balance

Classical mechanics places some restrictions on the form of the kernel $w\big(\Gamma,\Gamma_1\,\big|\,\Gamma',\Gamma_1'\big)$. In particular, if the microscopic dynamics are time-reversal symmetric and $\Gamma^{\mathcal{T}} = (-\boldsymbol{p},-\boldsymbol{L})$ denotes the kinematic variables under time reversal ($\mathcal{T}$), then

$$w\big(\Gamma',\Gamma_1'\,\big|\,\Gamma,\Gamma_1\big) = w\big(\Gamma^{\mathcal{T}},\Gamma_1^{\mathcal{T}}\,\big|\,\Gamma'^{\mathcal{T}},\Gamma_1'^{\mathcal{T}}\big) \quad . \tag{8.45}$$

This is because the time reverse of the process $|\Gamma,\Gamma_1\rangle \to |\Gamma',\Gamma_1'\rangle$ is $|\Gamma'^{\mathcal{T}},\Gamma_1'^{\mathcal{T}}\rangle \to |\Gamma^{\mathcal{T}},\Gamma_1^{\mathcal{T}}\rangle$.

We first consider equilibrium in the rest frame, where the equilibrium distribution is invariant under time reversal. Then in equilibrium we must have

$$w\big(\Gamma',\Gamma_1'\,\big|\,\Gamma,\Gamma_1\big)\; f^0(\Gamma)\, f^0(\Gamma_1)\, d^4\Gamma = w\big(\Gamma^{\mathcal{T}},\Gamma_1^{\mathcal{T}}\,\big|\,\Gamma'^{\mathcal{T}},\Gamma_1'^{\mathcal{T}}\big)\, f^0(\Gamma'^{\mathcal{T}})\, f^0(\Gamma_1'^{\mathcal{T}})\, d^4\Gamma^{\mathcal{T}} \tag{8.46}$$

where

$$d^4\Gamma \equiv d\Gamma\, d\Gamma_1\, d\Gamma' d\Gamma_1' \qquad , \qquad d^4\Gamma^{\mathcal{T}} \equiv d\Gamma^{\mathcal{T}}\, d\Gamma_1^{\mathcal{T}}\, d\Gamma'^{\mathcal{T}} d\Gamma_1'^{\mathcal{T}} \quad . \tag{8.47}$$

Since $d\Gamma = d\Gamma^{\mathcal{T}}$ *etc.*, we may cancel the differentials above, and after invoking eqn. 8.45 and suppressing the common $\boldsymbol{r}$ label, we find

$$f^0(\Gamma)\, f^0(\Gamma_1) = f^0(\Gamma'^{\mathcal{T}})\, f^0(\Gamma_1'^{\mathcal{T}}) \quad . \tag{8.48}$$

This is the condition of *detailed balance*. For the Boltzmann distribution, we have $f^0(\Gamma) = A\, e^{-\varepsilon/k_{\rm B}T}$, where A is a constant and where $\varepsilon = \varepsilon(\Gamma)$ is the kinetic energy, *e.g.* $\varepsilon(\Gamma) = \boldsymbol{p}^2/2m$ in the case of point particles. Note that $\varepsilon(\Gamma^{\mathcal{T}}) = \varepsilon(\Gamma)$. Detailed balance is satisfied because the kinematics of the collision requires energy conservation:

$$\varepsilon + \varepsilon_1 = \varepsilon' + \varepsilon_1' \quad . \tag{8.49}$$

[4]Recall from classical mechanics the definition of the *Poisson bracket*, $\{A,B\} = \partial_{\boldsymbol{r}} A\cdot\partial_{\boldsymbol{p}} B - \partial_{\boldsymbol{r}} B\cdot\partial_{\boldsymbol{p}} A$. Then from Hamilton's equations $\dot{\boldsymbol{r}} = \partial H/\partial\boldsymbol{p}$ and $\dot{\boldsymbol{p}} = -\partial H/\partial\boldsymbol{r}$, where $H(\boldsymbol{p},\boldsymbol{r},t)$ is the Hamiltonian, and thus we have $dA/dt = \{A,H\}$. Invariants have zero Poisson bracket with the Hamiltonian.

Since momentum is also kinematically conserved, *i.e.*

$$\boldsymbol{p} + \boldsymbol{p}_1 = \boldsymbol{p}' + \boldsymbol{p}_1' \quad , \tag{8.50}$$

any distribution of the form

$$f^0(\Gamma) = A\,\exp\!\left(-\,\frac{\varepsilon(\Gamma) - \boldsymbol{V}\cdot\boldsymbol{p} - \mu_{\boldsymbol{V}}}{k_{\rm B}T}\right) \tag{8.51}$$

also satisfies detailed balance, for any velocity parameter $\boldsymbol{V}$ and $f^0(\Gamma)f^0(\Gamma_1) = f^0(\Gamma')f^0(\Gamma_1')$. This distribution is appropriate for gases which are flowing with average particle velocity $\boldsymbol{V}$. For point particles, $A = h^{-d}$. Note that

$$\frac{\partial}{\partial \boldsymbol{p}}\big(\varepsilon - \boldsymbol{V}\cdot\boldsymbol{p}\big) = \boldsymbol{v} - \boldsymbol{V} \quad , \tag{8.52}$$

which is the velocity relative to the local flow. To obtain $\mu_{\boldsymbol{V}}$, define, for point particles,

$$\xi(\boldsymbol{V},T) \equiv \int d^d p\,\exp\!\left[-\,\frac{\varepsilon(\boldsymbol{p}) - \boldsymbol{V}\cdot\boldsymbol{p}}{k_{\rm B}T}\right] \quad . \tag{8.53}$$

Then at fixed number density n and temperature T we have

$$f^0(\boldsymbol{p}) = \frac{n}{\xi(\boldsymbol{V},T)}\exp\left[-\,\frac{\varepsilon(\boldsymbol{p}) - \boldsymbol{V}\cdot\boldsymbol{p}}{k_{\rm B}T}\right] \tag{8.54}$$

and

$$\mu_{\boldsymbol{V}} = \mu - k_{\rm B}T\log\!\left(\frac{\xi(\boldsymbol{V},T)}{\xi(\boldsymbol{0},T)}\right) \quad , \tag{8.55}$$

with $\mu = \mu_{\boldsymbol{V}=\boldsymbol{0}}$. For the ballistic dispersion $\varepsilon(\boldsymbol{p}) = \boldsymbol{p}^2/2m$, we obtain $\mu_{\boldsymbol{V}} = \mu - \frac{1}{2}m\boldsymbol{V}^2$.

In addition to time-reversal, parity is also a symmetry of the microscopic mechanical laws. Under the parity operation $\mathcal{P}$, we have $\boldsymbol{r} \to -\boldsymbol{r}$ and $\boldsymbol{p} \to -\boldsymbol{p}$. Note that a pseudovector such as $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$ is unchanged under $\mathcal{P}$. Thus, $\Gamma^{\mathcal{P}} = (-\boldsymbol{p}, \boldsymbol{L})$. Under the combined operation of $\mathcal{C} = \mathcal{P}\mathcal{T}$, we have $\Gamma^{\mathcal{C}} = (\boldsymbol{p}, -\boldsymbol{L})$. If the microscopic Hamiltonian is invariant under C, then we must have

$$w\big(\Gamma', \Gamma_1' \,\big|\, \Gamma, \Gamma_1\big) = w\big(\Gamma^{\mathcal{C}}, \Gamma_1^{\mathcal{C}} \,\big|\, \Gamma'^{\mathcal{C}}, \Gamma_1'^{\mathcal{C}}\big) \quad . \tag{8.56}$$

For point particles, invariance under T and P then means

$$w(\boldsymbol{p}', \boldsymbol{p}_1' \,|\, \boldsymbol{p}, \boldsymbol{p}_1) = w(\boldsymbol{p}, \boldsymbol{p}_1 \,|\, \boldsymbol{p}', \boldsymbol{p}_1') \quad , \tag{8.57}$$

and therefore the collision integral takes the simplified form,

$$\frac{Df(\boldsymbol{r},\boldsymbol{p},t)}{Dt} = \int d^3p_1 \int d^3p' \int d^3p_1'\, w(\boldsymbol{p}', \boldsymbol{p}_1' \,|\, \boldsymbol{p}, \boldsymbol{p}_1)\Big\{f(\boldsymbol{r},\boldsymbol{p}',t)\,f(\boldsymbol{r},\boldsymbol{p}_1',t) - f(\boldsymbol{r},\boldsymbol{p},t)\,f(\boldsymbol{r},\boldsymbol{p}_1,t)\Big\} = \left(\frac{df}{dt}\right)_{\rm coll} . \tag{8.58}$$

The most general statement of detailed balance is

$$\frac{f^0(\Gamma')\,f^0(\Gamma_1')}{f^0(\Gamma)\,f^0(\Gamma_1)} = \frac{w\big(\Gamma', \Gamma_1' \,\big|\, \Gamma, \Gamma_1\big)}{w\big(\Gamma, \Gamma_1 \,\big|\, \Gamma', \Gamma_1'\big)} \quad . \tag{8.59}$$

Under this condition, the collision term vanishes for $f = f^0$, which is the equilibrium distribution.

Detailed balance and single particle scattering

For single particle scattering, similar considerations lead to

$$f^0(\Gamma) = f^0(\Gamma'^{\mathcal{T}}) \quad , \quad \frac{f^0(\Gamma')}{f^0(\Gamma)} = \frac{w(\Gamma'|\Gamma)}{w(\Gamma|\Gamma')} \quad . \tag{8.60}$$

In the first of these equations, the time reversal operation also acts on the external potential off which the particles scatter. In general, detailed balance pertains when the static equilibrium distribution $f^0(\Gamma)$ is such that the integrand of the collision term vanishes for all Γ, hence $(df^0(\Gamma)/dt)_{\rm coll} = 0$.

8.3.6 Kinematics and cross section

We can rewrite eqn. 8.58 in the form

$$\frac{Df(\boldsymbol{r},\boldsymbol{p},t)}{Dt} = \int d^3p_1 \int d\Omega\, |\boldsymbol{v}-\boldsymbol{v}_1|\, \frac{d\sigma}{d\Omega}\Big\{ f(\boldsymbol{r},\boldsymbol{p}',t)\, f(\boldsymbol{r},\boldsymbol{p}_1',t) - f(\boldsymbol{r},\boldsymbol{p},t)\, f(\boldsymbol{r},\boldsymbol{p}_1,t)\Big\} \quad , \tag{8.61}$$

where $d\sigma/d\Omega$ is the *differential scattering cross section*. Recall the definition from scattering theory in classical mechanics,

$$d\sigma = \frac{\text{\# of particles scattered into solid angle } d\Omega \text{ per unit time}}{\text{incident flux}} \quad . \tag{8.62}$$

If we recast the scattering problem in terms of center-of-mass and relative coordinates, we conclude that the total momentum is conserved by the collision, and furthermore that the energy in the CM frame is conserved, which means that the magnitude of the *relative* momentum is conserved. Thus, we may write $\boldsymbol{p}' - \boldsymbol{p}_1' = |\boldsymbol{p}-\boldsymbol{p}_1|\,\hat{\boldsymbol{\Omega}}$, where $\hat{\boldsymbol{\Omega}}$ is a unit vector. Then $\boldsymbol{p}'$ and $\boldsymbol{p}_1'$ are determined to be

$$\begin{aligned} \boldsymbol{p}' &= \tfrac{1}{2}\big(\boldsymbol{p}+\boldsymbol{p}_1 + |\boldsymbol{p}-\boldsymbol{p}_1|\,\hat{\boldsymbol{\Omega}}\big) \\ \boldsymbol{p}_1' &= \tfrac{1}{2}\big(\boldsymbol{p}+\boldsymbol{p}_1 - |\boldsymbol{p}-\boldsymbol{p}_1|\,\hat{\boldsymbol{\Omega}}\big) \quad . \end{aligned} \tag{8.63}$$

Recall that for the scattering of classical hard spheres of radius a, the differential scattering cross section is $\frac{d\sigma}{d\Omega} = a^2$. Thus, the total scattering cross section is $\sigma_{\rm tot} = 4\pi a^2 = \pi d^2$, where $d = 2a$ is the sphere diameter. For Coulomb scattering of two point particles of charge q and mass m, one has

$$\frac{d\sigma}{d\Omega} = \left(\frac{2m_{\rm red}q^2}{|\boldsymbol{p}-\boldsymbol{p}_1|^2\,\sin^2(\frac{1}{2}\vartheta)}\right)^2 \quad , \tag{8.64}$$

where $m_{\rm red} = m/2$ is the reduced mass and $\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\Omega}} = \cos\vartheta$ where $\hat{\boldsymbol{n}} \equiv (\boldsymbol{p}-\boldsymbol{p}_1)/|\boldsymbol{p}-\boldsymbol{p}_1|$. The total cross section for Coulomb scattering diverges since the differential cross section behaves as ϑ^{-4} as $\vartheta \to 0$.

8.3.7 $\mathcal{H}$-theorem

To peek ahead, we are about to prove the following. Let

$$\begin{aligned} h(\boldsymbol{r},t) &= \int d^3p\, f(\boldsymbol{r},\boldsymbol{p},t)\, \log\bigl[f(\boldsymbol{r},\boldsymbol{p},t)\mathsf{a}^3\bigr] \\ \boldsymbol{j}(\boldsymbol{r},t) &= \int d^3p\, f(\boldsymbol{r},\boldsymbol{p},t)\, \log\bigl[f(\boldsymbol{r},\boldsymbol{p},t)\mathsf{a}^3\bigr]\, \frac{d\boldsymbol{r}}{dt} \quad . \end{aligned} \tag{8.65}$$

Here a can be any constant which has the appropriate dimensions of action. Then if $f(\boldsymbol{r},\boldsymbol{p},t)$ evolves according to the Boltzmann equation, it is necessarily the case that

$$\frac{\partial h(\boldsymbol{r},t)}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}(\boldsymbol{r},t) \le 0 \quad , \tag{8.66}$$

where $\boldsymbol{\nabla} \equiv \partial/\partial\boldsymbol{r}$. If we integrate over all space, and we adopt boundary conditions where $\boldsymbol{j} \to 0$ at spatial infinity,

$$\mathcal{H}(t) = \int d^3r\, h(\boldsymbol{r},t) \quad \Rightarrow \quad \frac{d\mathcal{H}}{dt} \le 0 \quad . \tag{8.67}$$

Thus, Boltzmann dynamics recognizes an arrow of time. Time increases in the direction that $\mathcal{H}$ decreases.

Let's consider the Boltzmann equation with two particle collisions. We define the local (*i.e.* $\boldsymbol{r}$-dependent) quantity

$$\rho_\varphi(\boldsymbol{r},t) \equiv \int d^3p\, f(\boldsymbol{r},\boldsymbol{p},t)\, \varphi\bigl(f(\boldsymbol{r},\boldsymbol{p},t)\bigr) \quad , \tag{8.68}$$

where $f = f(\boldsymbol{r},\boldsymbol{p},t)$ and $\varphi(f)$ is arbitrary. We now compute

$$\begin{aligned} \frac{\partial\rho_\varphi}{\partial t} &= \int d^3p\, \frac{\partial(f\varphi)}{\partial t} = \int d^3p\, \frac{\partial(f\varphi)}{\partial f}\,\frac{\partial f}{\partial t} \\ &= \int d^3p\, \frac{\partial(f\varphi)}{\partial f}\left\{ -\dot{\boldsymbol{r}}\cdot\frac{\partial f}{\partial\boldsymbol{r}} - \dot{\boldsymbol{p}}\cdot\frac{\partial f}{\partial\boldsymbol{p}} + \left(\frac{df}{dt}\right)_{\!\rm coll}\right\} \\ &= \int d^3p \left\{ -\dot{\boldsymbol{r}}\cdot\frac{\partial(f\varphi)}{\partial\boldsymbol{r}} - \dot{\boldsymbol{p}}\cdot\frac{\partial(f\varphi)}{\partial\boldsymbol{p}} + \frac{\partial(f\varphi)}{\partial f}\left(\frac{df}{dt}\right)_{\!\rm coll}\right\} \end{aligned} \tag{8.69}$$

We may integrate the second term in the brackets by parts on $\boldsymbol{p}$. Assuming $f = 0$ for infinite values of the kinematic variables, which is the only physical possibility, we then have

$$\begin{aligned} \frac{\partial\rho_\varphi}{\partial t} &= \int d^3p \left\{ -\dot{\boldsymbol{r}}\cdot\frac{\partial(f\varphi)}{\partial\boldsymbol{r}} - \frac{\partial\dot{\boldsymbol{r}}}{\partial\boldsymbol{r}}\,(f\varphi) + \frac{\partial(f\varphi)}{\partial f}\left(\frac{df}{dt}\right)_{\!\rm coll}\right\} \\ &= -\frac{\partial}{\partial\boldsymbol{r}}\int d^3p\, f\varphi\,\dot{\boldsymbol{r}} + \int d^3p\, \frac{\partial(f\varphi)}{\partial f}\left(\frac{df}{dt}\right)_{\!\rm coll} \quad . \end{aligned} \tag{8.70}$$

Thus,

$$\frac{\partial\rho_\varphi(\boldsymbol{r},t)}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}_\varphi(\boldsymbol{r},t) = \sigma_\varphi(\boldsymbol{r},t) \quad , \tag{8.71}$$

where

$$\begin{aligned}\boldsymbol{j}_\varphi(\boldsymbol{r},t) &= \int d^3p\, f(\boldsymbol{r},\boldsymbol{p},t)\,\varphi\big(f(\boldsymbol{r},\boldsymbol{p},t)\big)\,\boldsymbol{v}(\boldsymbol{p}) \\ \sigma_\varphi(\boldsymbol{r},t) &= \int d^3p\,\frac{\partial(f\varphi)}{\partial f}\bigg|_{f(\boldsymbol{r},\boldsymbol{p},t)}\left(\frac{df}{dt}\right)_{\rm coll}\end{aligned} \tag{8.72}$$

and $\dot{\boldsymbol{r}} = \boldsymbol{v}(\boldsymbol{p}) = \partial H_0/\partial\boldsymbol{p}$ is the velocity.

Thus, we arrive at eqn. 8.71, which is a continuity equation with a source term $\sigma_\varphi(\boldsymbol{r},t)$. The source term is nonzero only in the presence of collisions. We now evaluate σ_φ under the assumption that f satisfies the Boltzmann equation with two particle scattering. Thus,

$$\begin{aligned}\sigma_\varphi(\boldsymbol{r},t) &= \int d^3p\int d^3p_1\int d^3p'\int d^3p'_1\,\Big\{w\big(\boldsymbol{p},\boldsymbol{p}_1\,|\,\boldsymbol{p}',\boldsymbol{p}'_1\big)\,f(\boldsymbol{p}')f(\boldsymbol{p}'_1) - w\big(\boldsymbol{p}',\boldsymbol{p}'_1\,|\,\boldsymbol{p},\boldsymbol{p}_1\big)\,f(\boldsymbol{p})f(\boldsymbol{p}_1)\Big\}\chi(\boldsymbol{p}) \\ &= \int d^3p\int d^3p_1\int d^3p'\int d^3p'_1\,w\big(\boldsymbol{p}',\boldsymbol{p}'_1\,|\,\boldsymbol{p},\boldsymbol{p}_1\big)\,f(\boldsymbol{p})f(\boldsymbol{p}_1)\,\big(\chi(\boldsymbol{p}')-\chi(\boldsymbol{p})\big)\quad,\end{aligned} \tag{8.73}$$

where

$$\chi = \frac{\partial(f\varphi)}{\partial f} = \varphi + f\,\frac{\partial\varphi}{\partial f}\quad, \tag{8.74}$$

and where we have suppressed the $\boldsymbol{r}$ and t dependences. We now invoke the symmetry

$$w\big(\boldsymbol{p}',\boldsymbol{p}'_1\,|\,\boldsymbol{p},\boldsymbol{p}_1\big) = w\big(\boldsymbol{p}'_1,\boldsymbol{p}'\,|\,\boldsymbol{p}_1,\boldsymbol{p}\big)\quad, \tag{8.75}$$

which allows us to write

$$\sigma = \tfrac{1}{2}\int d^3p\int d^3p_1\int d^3p'\int d^3p'_1\,w\big(\boldsymbol{p}',\boldsymbol{p}'_1\,|\,\boldsymbol{p},\boldsymbol{p}_1\big)\,f(\boldsymbol{p})\,f(\boldsymbol{p}_1)\,\big(\chi(\boldsymbol{p}')+\chi(\boldsymbol{p}'_1)-\chi(\boldsymbol{p})-\chi(\boldsymbol{p}_1)\big)\quad, \tag{8.76}$$

This shows that $\sigma_\varphi = 0$ if $\chi(\boldsymbol{p})$ is a collisional invariant, *i.e.* if $\chi(\boldsymbol{p})+\chi(\boldsymbol{p}_1) = \chi(\boldsymbol{p}')+\chi(\boldsymbol{p}'_1)$ for every allowed collision.

Now let us fix $\varphi(f) = \log(f\mathsf{a}^3)$ and evaluate the source term $\sigma \equiv \sigma_{\varphi=\log(f\mathsf{a}^3)}$. We have

$$\sigma = -\tfrac{1}{2}\int d^3p\int d^3p_1\int d^3p'\int d^3p'_1\,w\big(\boldsymbol{p}',\boldsymbol{p}'_1\,|\,\boldsymbol{p},\boldsymbol{p}_1\big)\,f(\boldsymbol{p}')f(\boldsymbol{p}'_1)\cdot x(\boldsymbol{p},\boldsymbol{p}_1\,|\,\boldsymbol{p}',\boldsymbol{p}'_1)\log x(\boldsymbol{p},\boldsymbol{p}_1\,|\,\boldsymbol{p}',\boldsymbol{p}'_1)\quad, \tag{8.77}$$

where $x(\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}',\boldsymbol{p}'_1) \equiv f(\boldsymbol{p})f(\boldsymbol{p}_1)/f(\boldsymbol{p}')f(\boldsymbol{p}'_1)$. We next invoke the result

$$\int d^3p'\int d^3p'_1\,w\big(\boldsymbol{p}',\boldsymbol{p}'_1\,|\,\boldsymbol{p},\boldsymbol{p}_1\big) = \int d^3p'\int d^3p'_1\,w\big(\boldsymbol{p},\boldsymbol{p}_1\,|\,\boldsymbol{p}',\boldsymbol{p}'_1\big) \tag{8.78}$$

which is a statement of unitarity of the scattering matrix[5]. Multiplying both sides by $f(\boldsymbol{p})\,f(\boldsymbol{p}_1)$, then integrating over $\boldsymbol{p}$ and $\boldsymbol{p}_1$, and finally changing variables $(\boldsymbol{p},\boldsymbol{p}_1)\leftrightarrow(\boldsymbol{p}',\boldsymbol{p}'_1)$, we find

$$\begin{aligned}0 &= \int d^3p\int d^3p_1\int d^3p'\int d^3p'_1\,w\big(\boldsymbol{p}',\boldsymbol{p}'_1\,|\,\boldsymbol{p},\boldsymbol{p}_1\big)\,\Big(f(\boldsymbol{p})f(\boldsymbol{p}_1)-f(\boldsymbol{p}')f(\boldsymbol{p}'_1)\Big) \\ &= \int d^3p\int d^3p_1\int d^3p'\int d^3p'_1\,w\big(\boldsymbol{p}',\boldsymbol{p}'_1\,|\,\boldsymbol{p},\boldsymbol{p}_1\big)\,f(\boldsymbol{p}')f(\boldsymbol{p}'_1)\,\Big\{x(\boldsymbol{p},\boldsymbol{p}_1\,|\,\boldsymbol{p}',\boldsymbol{p}'_1)-1\Big\}\quad.\end{aligned} \tag{8.79}$$

[5]See Lifshitz and Pitaevskii, *Physical Kinetics*, §2.

Multiplying this result by $\frac{1}{2}$ and adding it to the previous equation for $\dot{h}$, we arrive at our final result,

$$\sigma = -\frac{1}{2}\int d^3p \int d^3p_1 \int d^3p' \int d^3p'_1 \, w(\boldsymbol{p}', \boldsymbol{p}'_1 \,|\, \boldsymbol{p}, \boldsymbol{p}_1)\, f(\boldsymbol{p}')f(\boldsymbol{p}'_1)\, (x \log x - x + 1) \quad , \tag{8.80}$$

where $x \equiv x(\boldsymbol{p}, \boldsymbol{p}_1; \boldsymbol{p}', \boldsymbol{p}'_1) = f(\boldsymbol{p})f(\boldsymbol{p}_1)/f(\boldsymbol{p}')f(\boldsymbol{p}'_1)$. It is now easy to prove that the function $g(x) = x \log x - x + 1$ is nonnegative for all positive x values[6], which therefore entails the important result

$$\frac{\partial h(\boldsymbol{r}, t)}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j}(\boldsymbol{r}, t) = \sigma(\boldsymbol{r}, t) \le 0 \quad . \tag{8.81}$$

Boltzmann's $\mathcal{H}$ function is the space integral of the local density $h(\boldsymbol{r})$: $\mathcal{H} = \int d^3r \, h(\boldsymbol{r})$.

Thus, everywhere in space, the source term $\sigma(\boldsymbol{r}, t)$ is nonpositive. In thermodynamic equilibrium the entropy production vanishes, so $\sigma = 0$. Equation 8.80 then requires $x = 1$ for every allowed collision with nonzero statistical weight. Thus,

$$f^0(\boldsymbol{p})\, f^0(\boldsymbol{p}_1) = f^0(\boldsymbol{p}')\, f^0(\boldsymbol{p}'_1) \quad , \tag{8.82}$$

or, taking the logarithm,

$$\log f^0(\boldsymbol{p}) + \log f^0(\boldsymbol{p}_1) = \log f^0(\boldsymbol{p}') + \log f^0(\boldsymbol{p}'_1) \quad . \tag{8.83}$$

But this means that $\log f^0$ is itself a collisional invariant, and if 1, $\boldsymbol{p}$, and ε are the only collisional invariants, then $\log f^0$ must be expressible in terms of them. Thus, we may write

$$-\log\left[h^3 f^0(\boldsymbol{p})\right] = \frac{\varepsilon(\boldsymbol{p}) - \boldsymbol{V}\cdot\boldsymbol{p} - \mu_{\boldsymbol{V}}}{k_{\rm B}T} \quad , \tag{8.84}$$

with $\mu_{\boldsymbol{V}}$, $\boldsymbol{V}$, and T constants which parameterize the equilibrium distribution $f^0(\boldsymbol{p})$, corresponding to the chemical potential, flow velocity, and temperature, respectively, and where $\mu_{\boldsymbol{V}}$ is related to the $\boldsymbol{V} = 0$ chemical potential μ according to eqn. 8.55.

8.4 Weakly Inhomogeneous Gas

Consider a gas which is only weakly out of equilibrium. We follow the treatment in Lifshitz and Pitaevskii, §6. As the gas is only slightly out of equilibrium, we seek a solution to the Boltzmann equation of the form $f = f^0 + \delta f$, where f^0 describes a local equilibrium. Recall that such a distribution function is annihilated by the collision term in the Boltzmann equation but not by the streaming term, hence a correction δf must be added in order to obtain a solution. The Boltzmann equation is written

$$\left(\frac{\partial}{\partial t} + \frac{\boldsymbol{p}}{m}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)(f^0 + \delta f) = \left(\frac{df}{dt}\right)_{\rm coll} \quad . \tag{8.85}$$

[6]The function $g(x) = x\log x - x + 1$ satisfies $g'(x) = \log x$, hence $g'(x) < 0$ on the interval $x \in [0, 1)$ and $g'(x) > 0$ on $x \in (1, \infty]$. Thus, $g(x)$ monotonically decreases from $g(0) = 1$ to $g(1) = 0$, and then monotonically increases to $g(\infty) = \infty$, never becoming negative.

The RHS of this equation must be of order δf because the local equilibrium distribution f^0 is annihilated by the collision integral.

To first order about a state at rest, the local equilibrium distribution may be written as

$$f^0(\boldsymbol{r},\Gamma) = C\exp\left(\frac{\mu-\varepsilon(\Gamma)+\boldsymbol{V}\cdot\boldsymbol{p}}{k_{\rm B}T}\right) \quad , \tag{8.86}$$

where $\mu=\mu(\boldsymbol{r},t)$, $T=T(\boldsymbol{r},t)$, and $\boldsymbol{V}=\boldsymbol{V}(\boldsymbol{r},t)$ may all vary in both space and time. Note that

$$\begin{aligned} df^0 &= \left(d\mu + \boldsymbol{p}\cdot d\boldsymbol{V} + (\varepsilon-\boldsymbol{V}\cdot\boldsymbol{p}-\mu)\,\frac{dT}{T} - d\varepsilon\right)\left(-\frac{\partial f^0}{\partial\varepsilon}\right) \\ &= \left(\frac{1}{n}\,dp + \boldsymbol{p}\cdot d\boldsymbol{V} + (\varepsilon-h)\,\frac{dT}{T} - d\varepsilon\right)\left(-\frac{\partial f^0}{\partial\varepsilon}\right) \quad , \end{aligned} \tag{8.87}$$

where $h=\mu+Ts$ is the enthalpy per particle, and where we drop the term proportional to $\boldsymbol{V}\cdot\boldsymbol{p}\,dT$, which on the assumption that $\boldsymbol{V}=0$ on average is second order in smallness. We have also invoked local thermodynamics in the form

$$d\mu = \left(\frac{\partial\mu}{\partial T}\right)_p dT + \left(\frac{\partial\mu}{\partial p}\right)_T dp = -s\,dT + \frac{1}{n}\,dp \quad , \tag{8.88}$$

where s is the entropy per particle and n is the number density. Note as well that

$$-\frac{\partial f^0}{\partial\varepsilon} = \frac{f^0}{k_{\rm B}T} \quad , \tag{8.89}$$

which we shall invoke further on below.

We wish to evaluate one of the contributions to the LHS of eqn. 8.85:

$$\begin{aligned} \frac{\partial f^0}{\partial t} + \frac{\boldsymbol{p}}{m}\cdot\frac{\partial f^0}{\partial\boldsymbol{r}} + \boldsymbol{F}\cdot\frac{\partial f^0}{\partial\boldsymbol{p}} = \Bigg\{ &\frac{\varepsilon-h}{T}\,\frac{\partial T}{\partial t} + m\boldsymbol{v}\cdot\Big[(\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{V}\Big] + \boldsymbol{v}\cdot\left(m\,\frac{\partial\boldsymbol{V}}{\partial t} + \frac{1}{n}\,\boldsymbol{\nabla}p\right) \\ &+ \frac{1}{n}\,\frac{\partial p}{\partial t} + \frac{\varepsilon-h}{T}\,\boldsymbol{v}\cdot\boldsymbol{\nabla}T - \boldsymbol{F}\cdot\boldsymbol{v}\Bigg\}\left(-\frac{\partial f^0}{\partial\varepsilon}\right) \quad . \end{aligned} \tag{8.90}$$

To simplify this, first note that Newton's laws applied to an ideal fluid give $\rho\dot{\boldsymbol{V}} = -\boldsymbol{\nabla}p + n\boldsymbol{F}$, where $\rho=mn$ is the mass density. Corrections to this result, *e.g.* viscosity and nonlinearity in $\boldsymbol{V}$, are of higher order.

Next, continuity for particle number means $\dot{n}+\boldsymbol{\nabla}\cdot(n\boldsymbol{V})=0$. We assume $\boldsymbol{V}$ is zero on average and that all derivatives are small, hence $\boldsymbol{\nabla}\cdot(n\boldsymbol{V}) = \boldsymbol{V}\cdot\boldsymbol{\nabla}n + n\,\boldsymbol{\nabla}\cdot\boldsymbol{V} \approx n\,\boldsymbol{\nabla}\cdot\boldsymbol{V}$. Thus,

$$\frac{\partial\log n}{\partial t} = \frac{\partial\log p}{\partial t} - \frac{\partial\log T}{\partial t} = -\boldsymbol{\nabla}\cdot\boldsymbol{V} \quad , \tag{8.91}$$

where we have invoked the ideal gas law $n=p/k_{\rm B}T$ above.

Next, we invoke conservation of entropy. If s is the entropy per particle, then ns is the entropy per unit volume, in which case we have the continuity equation

$$\frac{\partial(ns)}{\partial t} + \boldsymbol{\nabla}\cdot(ns\boldsymbol{V}) = n\left(\frac{\partial s}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}s\right) + s\left(\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\boldsymbol{V})\right) = 0 \quad . \tag{8.92}$$

The second bracketed term on the RHS vanishes because of particle continuity, leaving us with the combination $\dot{s} + \boldsymbol{V}\cdot\boldsymbol{\nabla}s \approx \dot{s} = 0$ (since $\boldsymbol{V} = 0$ on average, and any gradient is first order in smallness). Now thermodynamics says

$$ds = \left(\frac{\partial s}{\partial T}\right)_p dT + \left(\frac{\partial s}{\partial p}\right)_T dp = \frac{c_p}{T}\,dT - \frac{k_{\rm B}}{p}\,dp \quad , \tag{8.93}$$

since $c_p = T\big(\partial s/\partial T\big)_p$ and $\big(\partial s/\partial p\big)_T = -\big(\partial v/\partial T\big)_p$, where $v = V/N$. Here, c_p is the heat capacity per particle at constant pressure[7]. Thus,

$$\frac{c_p}{k_{\rm B}}\,\frac{\partial\log T}{\partial t} - \frac{\partial\log p}{\partial t} = 0 \quad . \tag{8.94}$$

We now have in eqns. 8.91 and 8.94 two equations in the two unknowns $\partial\log T/\partial t$ and $\partial\log p/\partial t$, yielding

$$\frac{\partial\log T}{\partial t} = -\frac{k_{\rm B}}{c_V}\,\boldsymbol{\nabla}\cdot\boldsymbol{V} \qquad , \qquad \frac{\partial\log p}{\partial t} = -\frac{c_p}{c_V}\,\boldsymbol{\nabla}\cdot\boldsymbol{V} \quad . \tag{8.95}$$

Thus eqn. 8.90 becomes

$$\frac{\partial f^0}{\partial t} + \frac{\boldsymbol{p}}{m}\cdot\frac{\partial f^0}{\partial\boldsymbol{r}} + \boldsymbol{F}\cdot\frac{\partial f^0}{\partial\boldsymbol{p}} = \left(-\frac{\partial f^0}{\partial\varepsilon}\right)\left\{\frac{\varepsilon-h}{T}\,\boldsymbol{v}\cdot\boldsymbol{\nabla}T + m\,v_\alpha v_\beta\,\mathcal{Q}_{\alpha\beta} + \frac{h - Tc_p - \varepsilon}{c_V/k_{\rm B}}\,\boldsymbol{\nabla}\cdot\boldsymbol{V}\right\} \quad , \tag{8.96}$$

where $\varepsilon = \varepsilon(\Gamma)$ and

$$\mathcal{Q}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial V_\alpha}{\partial x_\beta} + \frac{\partial V_\beta}{\partial x_\alpha}\right) \quad . \tag{8.97}$$

Therefore, the Boltzmann equation takes the form

$$\frac{\partial\,\delta f}{\partial t} + \left\{\frac{\varepsilon(\Gamma)-h}{T}\,\boldsymbol{v}\cdot\boldsymbol{\nabla}T + m\,v_\alpha v_\beta\,\mathcal{Q}_{\alpha\beta} - \frac{\varepsilon(\Gamma)-h+Tc_p}{c_V/k_{\rm B}}\,\boldsymbol{\nabla}\cdot\boldsymbol{V}\right\}\frac{f^0}{k_{\rm B}T} = \left(\frac{df}{dt}\right)_{\rm coll} \quad . \tag{8.98}$$

Notice we have dropped the terms $\boldsymbol{v}\cdot\partial\,\delta f/\partial\boldsymbol{r}$ and $\boldsymbol{F}\cdot\partial\,\delta f/\partial\boldsymbol{p}$, since δf must already be first order in smallness, and both the $\partial/\partial\boldsymbol{r}$ operator as well as $\boldsymbol{F}$ add a second order of smallness, which is negligible. Typically $\partial\,\delta f/\partial t$ is nonzero if the applied force $\boldsymbol{F}(t)$ is time-dependent. We use the convention of summing over repeated indices. Note that $\delta_{\alpha\beta}\,\mathcal{Q}_{\alpha\beta} = \mathcal{Q}_{\alpha\alpha} = \boldsymbol{\nabla}\cdot\boldsymbol{V}$. For ideal gases in which only translational and rotational degrees of freedom are excited, $h = c_pT$.

[7] In the chapter on thermodynamics, we adopted a slightly different definition of c_p as the heat capacity per mole. In this chapter c_p is the heat capacity per particle.

8.5 Relaxation Time Approximation

8.5.1 Approximation of collision integral

We now consider a very simple model of the collision integral,

$$\left(\frac{df}{dt}\right)_{\rm coll} = -\frac{f - f^0}{\tau} = -\frac{\delta f}{\tau} \quad . \tag{8.99}$$

This model is known as the *relaxation time approximation*[8]. Here, $f^0 = f^0(\boldsymbol{r}, \boldsymbol{p}, t)$ is a distribution function which describes a *local equilibrium* at each position $\boldsymbol{r}$ and time t. The quantity τ is the *relaxation time*, which can in principle be momentum-dependent, but which we shall first consider to be constant. In the absence of streaming terms, we have, for a time-independent $f^0(\boldsymbol{p})$,

$$\frac{\partial\,\delta f}{\partial t} = -\frac{\delta f}{\tau} \qquad \Longrightarrow \qquad \delta f(\boldsymbol{r}, \boldsymbol{p}, t) = \delta f(\boldsymbol{r}, \boldsymbol{p}, 0)\, e^{-t/\tau} \quad . \tag{8.100}$$

The distribution f then relaxes to the equilibrium distribution f^0 on a time scale τ. We note that this approximation is obviously flawed in that all quantities – even the collisional invariants – relax to their equilibrium values on the scale τ. In the Appendix II, we consider a model for the collision integral in which the collisional invariants are all preserved, but everything else relaxes to local equilibrium at a single rate.

8.5.2 Computation of the scattering time

Consider two particles with velocities $\boldsymbol{v}$ and $\boldsymbol{v}'$. The average of their relative *speed* is

$$\langle\, |\boldsymbol{v} - \boldsymbol{v}'|\,\rangle = \int d^3v \int d^3v'\; P(\boldsymbol{v})\, P(\boldsymbol{v}')\, |\boldsymbol{v} - \boldsymbol{v}'| \quad , \tag{8.101}$$

where $P(\boldsymbol{v})$ is the Maxwell velocity distribution,

$$P(\boldsymbol{v}) = \left(\frac{m}{2\pi k_{\rm B} T}\right)^{3/2} \exp\left(-\frac{m\boldsymbol{v}^2}{2k_{\rm B}T}\right) \quad , \tag{8.102}$$

which follows from the Boltzmann form of the equilibrium distribution $f^0(\boldsymbol{p})$. It is left as an exercise for the student to verify that

$$\bar{v}_{\rm rel} \equiv \langle\, |\boldsymbol{v} - \boldsymbol{v}'|\,\rangle = \frac{4}{\sqrt{\pi}} \left(\frac{k_{\rm B}T}{m}\right)^{1/2} \quad . \tag{8.103}$$

Note that $\bar{v}_{\rm rel} = \sqrt{2}\,\bar{v}$, where $\bar{v}$ is the average particle speed. Let σ be the total scattering cross section, which for hard spheres is $\sigma = \pi d^2$, where d is the hard sphere diameter. Then the rate at which particles scatter is

$$\frac{1}{\tau} = n\,\bar{v}_{\rm rel}\,\sigma \quad . \tag{8.104}$$

[8] In the nonconserving version considered here, f^0 is a prescribed target rather than a Maxwellian matched to the evolving local conserved moments.

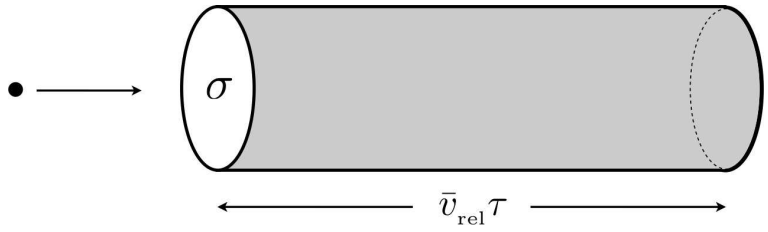


Figure 8.3: Graphic representation of the equation $n\,\sigma\,\bar{v}_{\rm rel}\,\tau = 1$, which yields the scattering time τ in terms of the number density n, average particle pair relative velocity $\bar{v}_{\rm rel}$, and two-particle total scattering cross section σ. The equation says that on average there must be one particle within the tube.

The particle *mean free path* is then

$$\ell = \bar{v}\,\tau = \frac{1}{\sqrt{2}\,n\,\sigma} \quad . \tag{8.105}$$

While the scattering length is not temperature-dependent within this formalism, the scattering time *is* T-dependent, with

$$\tau(T) = \frac{1}{n\,\bar{v}_{\rm rel}\,\sigma} = \frac{\sqrt{\pi}}{4n\sigma}\left(\frac{m}{k_{\rm B}T}\right)^{1/2} \tag{8.106}$$

at fixed number density. As $T \to 0$, the collision time diverges as $\tau \propto T^{-1/2}$, because the particles on average move more slowly at lower temperatures. The mean free path, however, is independent of T, and is given by $\ell = 1/\sqrt{2}n\sigma$.

8.5.3 Thermal conductivity

We consider a system with a temperature gradient $\boldsymbol{\nabla}T$ and seek a steady state (*i.e.* time-independent) solution to the Boltzmann equation. We restrict our attention to monatomic ideal gases with $\varepsilon(\boldsymbol{p}(= \boldsymbol{p}^2/2m$ and $c_p = \frac{5}{2}k_{\rm B}$. We further assume $F_\alpha = \mathcal{Q}_{\alpha\beta} = 0$. Appealing to eqn. 8.98, and using the relaxation time approximation for the collision integral, we have

$$\delta f = -\frac{\tau(\varepsilon - c_p\,T)}{k_{\rm B}T^2}\,(\boldsymbol{v}\cdot\boldsymbol{\nabla}T)\,f^0 \quad . \tag{8.107}$$

We are now ready to compute the energy and particle currents. In order to compute the *local density* of any quantity $A(\boldsymbol{r},\boldsymbol{p})$, we multiply by the distribution $f(\boldsymbol{r},\boldsymbol{p})$ and integrate over momentum:

$$\rho_A(\boldsymbol{r},t) = \int d^3p\,A(\boldsymbol{r},\boldsymbol{p})\,f(\boldsymbol{r},\boldsymbol{p},t) \quad , \tag{8.108}$$

For the energy (thermal) current, we let $A = \varepsilon\,v_\alpha = \varepsilon\,p_\alpha/m$, in which case $\rho_A = j_\alpha$. Note $\int d^3p\,\boldsymbol{p}\,f^0 = 0$ since f^0 is isotropic in $\boldsymbol{p}$ even when μ and T depend on $\boldsymbol{r}$. Thus, only δf enters into the calculation of the various currents. Thus, the energy (thermal) current is

$$j^\alpha_\varepsilon(\boldsymbol{r}) = \int d^3p\,\varepsilon\,v^\alpha\,\delta f = -\frac{n\tau}{k_{\rm B}T^2}\left\langle\, v^\alpha v^\beta\,\varepsilon\,(\varepsilon - c_p\,T)\,\right\rangle\frac{\partial T}{\partial x^\beta} \quad , \tag{8.109}$$

where the repeated index β is summed over, and where momentum averages are defined relative to the equilibrium distribution, *i.e.*

$$\langle\, \phi(\boldsymbol{p})\,\rangle = \int d^3p\, \phi(\boldsymbol{p})\, f^0(\boldsymbol{p}) \Big/ \int d^3p\, f^0(\boldsymbol{p}) = \int d^3v\, P(\boldsymbol{v})\, \phi(m\boldsymbol{v}) \quad . \tag{8.110}$$

In this context, it is useful to invoke the identity $d^3p\, f^0(\boldsymbol{p}) = n\, d^3v\, P(\boldsymbol{v})$, where

$$P(\boldsymbol{v}) = \left(\frac{m}{2\pi k_{\rm B} T}\right)^{3/2} e^{-m(\boldsymbol{v}-\boldsymbol{V})^2/2k_{\rm B}T} \tag{8.111}$$

is the Maxwell velocity distribution.

Note that if $\phi = \phi(\varepsilon)$ is a function of the energy, and if $\boldsymbol{V} = 0$, then[9]

$$d^3p\, f^0(\boldsymbol{p}) = n\, d^3v\, P(\boldsymbol{v}) = n\, \mathcal{P}(\varepsilon)\, d\varepsilon \quad , \tag{8.112}$$

where

$$\mathcal{P}(\varepsilon) = \frac{2}{\sqrt{\pi}}\, (k_{\rm B}T)^{-3/2}\, \varepsilon^{1/2}\, e^{-\varepsilon/k_{\rm B}T} \quad , \tag{8.113}$$

is the Maxwellian distribution expressed in terms of ε, and is normalized according to $\int_0^\infty d\varepsilon\, \mathcal{P}(\varepsilon) = 1$.[10] Averages with respect to this distribution are given by

$$\langle\, \phi(\varepsilon)\,\rangle = \int_0^\infty d\varepsilon\, \phi(\varepsilon)\, \mathcal{P}(\varepsilon) = \frac{2}{\sqrt{\pi}}\, (k_{\rm B}T)^{-3/2} \int_0^\infty d\varepsilon\, \varepsilon^{1/2}\, \phi(\varepsilon)\, e^{-\varepsilon/k_{\rm B}T} \quad . \tag{8.114}$$

For homogeneous $\phi(\varepsilon) = \varepsilon^s$ we have

$$\langle\, \varepsilon^s\,\rangle = \frac{2}{\sqrt{\pi}}\, \Gamma\big(s + \tfrac{3}{2}\big) (k_{\rm B}T)^s \quad , \tag{8.115}$$

with $\mathrm{Re}\, s > -\frac{3}{2}$ necessary for convergence. Due to spatial isotropy, it is clear that we can replace $v^\alpha\, v^\beta$ by $\frac{1}{3}\boldsymbol{v}^2\, \delta_{\alpha\beta}$ and then $\varepsilon = \frac{1}{2}m\boldsymbol{v}^2$ in eqn. 8.109. We then have $\boldsymbol{j}_\varepsilon = -\kappa\, \boldsymbol{\nabla} T$, with

$$\kappa = \frac{2n\tau}{3mk_{\rm B}T^2}\, \langle\, \varepsilon^2 \big(\varepsilon - c_p\, T\big)\,\rangle = \frac{5n\tau k_{\rm B}^2 T}{2m} = \frac{\pi}{8} n\ell\bar{v}\, c_p \quad , \tag{8.116}$$

where $c_p = \frac{5}{2}k_{\rm B}$ and $\bar{v}^2 = 8k_{\rm B}T/\pi m$. The quantity κ is called the *thermal conductivity*. Note that $\kappa \propto T^{1/2}$.

[9] Here we are being somewhat sloppy with notation and implicitly integrating over direction. Thus $d^3p = 4\pi p^2 dp$ and similarly $d^3v = 4\pi v^2 dv$.

[10] $\mathcal{P}(\varepsilon)$ can also be recognized as the normalized product of the Boltzmann energy distribution $\exp(-\varepsilon/k_{\rm B}T)$ and the density of states $g(\varepsilon)$. In $d = 3$ we have $g(\varepsilon) \propto \varepsilon^{1/2}$.

8.5.4 Viscosity

Consider the situation depicted in fig. 8.4. A fluid filling the space between two large flat plates at $z = 0$ and $z = d$ is set in motion by a force $\boldsymbol{F} = F\boldsymbol{x}$ applied to the upper plate; the lower plate is fixed. It is assumed that the fluid's velocity locally matches that of the plates. Fluid particles at the top have an average x-component of their momentum $\langle p_x \rangle = mV$. As these particles move downward toward lower z values, they bring their x-momenta with them. Therefore there is a downward ($-\hat{\boldsymbol{z}}$-directed) flow of $\langle p_x \rangle$. Since x-momentum is constantly being drawn away from $z = d$ plane, this means that there is a $-\boldsymbol{x}$-directed *viscous drag* on the upper plate. The viscous drag force per unit area is given by $F_{\rm drag}/A = -\eta V/d$, where $V/d = \partial V_x/\partial z$ is the velocity gradient and η is the *shear viscosity*. In steady state, the applied force balances the drag force, *i.e.* $F + F_{\rm drag} = 0$. Clearly in the steady state the spatially averaged momentum density of the fluid does not change, and is given by $\frac{1}{2}\rho V\boldsymbol{x}$, where ρ is the fluid mass density. The momentum per unit time injected into the fluid by the upper plate at $z = d$ is then extracted by the lower plate at $z = 0$. The *momentum flux density* $\Pi_{xz} = n\,\langle\, p_x\, v_z \,\rangle$ is the drag force on the upper surface per unit area: $\Pi_{xz} = -\eta\,\partial V_x/\partial z$. The units of viscosity are $[\eta] = M/LT$.

We now provide some formal definitions of viscosity. As we shall see presently, there is in fact a second type of viscosity, called second viscosity or *bulk viscosity*, which is measurable although not by the type of experiment depicted in fig. 8.4.

The momentum flux tensor $\Pi_{\alpha\beta} = n\,\langle\, p_\alpha\, v_\beta \,\rangle$ is defined to be the current of momentum component p_α in the direction of increasing x_β. For a gas in motion with average velocity $\boldsymbol{V}$, we have

$$\begin{aligned}\Pi_{\alpha\beta} &= nm\,\langle\,(V_\alpha + v'_\alpha)(V_\beta + v'_\beta)\,\rangle \\ &= nm\,V_\alpha V_\beta + nm\,\langle\, v'_\alpha v'_\beta \,\rangle \\ &= nm\,V_\alpha V_\beta + \tfrac{1}{3}nm\,\langle\, \boldsymbol{v}'^2 \,\rangle\,\delta_{\alpha\beta} = \rho\,V_\alpha V_\beta + p\,\delta_{\alpha\beta} \quad ,\end{aligned} \tag{8.117}$$

where $\boldsymbol{v}'$ is the particle velocity in a frame moving with velocity $\boldsymbol{V}$, and where we have invoked the ideal gas law $p = nk_{\rm B}T$. The mass density is $\rho = nm$.

When $\boldsymbol{V}$ is spatially varying,

$$\Pi_{\alpha\beta} = p\,\delta_{\alpha\beta} + \rho\,V_\alpha V_\beta - \tilde{\sigma}_{\alpha\beta} \quad , \tag{8.118}$$

where $\tilde{\sigma}_{\alpha\beta}$ is the *viscosity stress tensor*. Any symmetric tensor, such as $\tilde{\sigma}_{\alpha\beta}$, can be decomposed into a sum of (i) a traceless component, and (ii) a component proportional to the identity matrix. Since $\tilde{\sigma}_{\alpha\beta}$ should be, to first order, linear in the spatial derivatives of the components of the velocity field $\boldsymbol{V}$, there is a unique two-parameter decomposition:

$$\begin{aligned}\tilde{\sigma}_{\alpha\beta} &= \eta\left(\frac{\partial V_\alpha}{\partial x_\beta} + \frac{\partial V_\beta}{\partial x_\alpha} - \tfrac{2}{3}\,\boldsymbol{\nabla}\cdot\boldsymbol{V}\,\delta_{\alpha\beta}\right) + \zeta\,\boldsymbol{\nabla}\cdot\boldsymbol{V}\,\delta_{\alpha\beta} \\ &= 2\eta\left(\mathcal{Q}_{\alpha\beta} - \tfrac{1}{3}\,\mathsf{Tr}\,(\mathcal{Q})\,\delta_{\alpha\beta}\right) + \zeta\;\mathsf{Tr}\,(\mathcal{Q})\,\delta_{\alpha\beta} \quad ,\end{aligned} \tag{8.119}$$

where $\mathcal{Q}_{\alpha\beta}$ is defined in eqn. 8.97. The coefficient of the traceless component is η, known as the *shear viscosity*. The coefficient of the component proportional to the identity is ζ, known as the *bulk viscosity*. The full stress tensor $\sigma_{\alpha\beta}$ contains a contribution from the pressure:

$$\sigma_{\alpha\beta} = -p\,\delta_{\alpha\beta} + \tilde{\sigma}_{\alpha\beta} \quad . \tag{8.120}$$

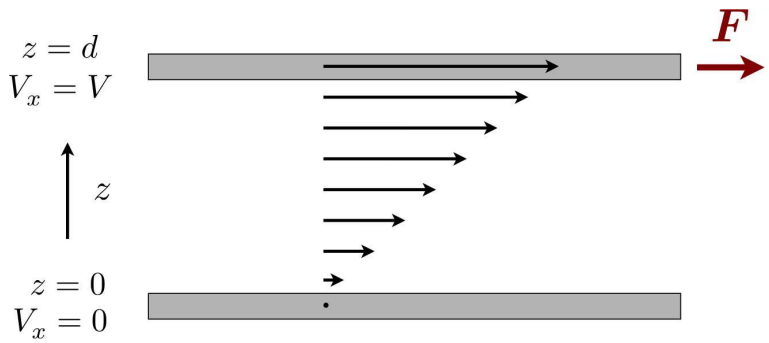


Figure 8.4: *Gedankenexperiment* to measure shear viscosity η in a fluid. The lower plate is fixed. The viscous drag force per unit area on the upper plate is $F_{\rm drag}/A = -\eta V/d$. This must be balanced by an applied force F.

The differential force dF_α that a fluid exerts on a surface element $\hat{\boldsymbol{n}}\, dA$ is $dF_\alpha = -\sigma_{\alpha\beta}\, n_\beta\, dA$, where we are using the Einstein summation convention and summing over the repeated index β. We will now compute the shear viscosity η using the Boltzmann equation in the relaxation time approximation.

Appealing again to eqn. 8.98, with $\boldsymbol{F} = 0$ and $h = c_p T$, we find

$$\delta f = -\frac{\tau}{k_{\rm B} T}\left\{ m\, v_\alpha v_\beta\, \mathcal{Q}_{\alpha\beta} + \frac{\varepsilon - c_p\, T}{T}\, \boldsymbol{v}\cdot\boldsymbol{\nabla} T - \frac{\varepsilon}{c_V/k_{\rm B}}\, \boldsymbol{\nabla}\cdot\boldsymbol{V}\right\} f^0 \quad . \tag{8.121}$$

We assume $\boldsymbol{\nabla} T = \boldsymbol{\nabla}\cdot\boldsymbol{V} = 0$, and we compute the momentum flux:

$$\begin{aligned}\Pi_{xz} &= \int d^3p\, p_x v_z\, \delta f = -\frac{nm^2\tau}{k_{\rm B} T}\, \mathcal{Q}_{\alpha\beta}\, \langle\, v_x\, v_z\, v_\alpha\, v_\beta\,\rangle \\ &= -\frac{n\tau}{k_{\rm B} T}\left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right)\langle\, m v_x^2 \cdot m v_z^2\,\rangle = -n\tau k_{\rm B} T\left(\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z}\right) \quad .\end{aligned} \tag{8.122}$$

Thus, if $V_x = V_x(z)$, we have

$$\Pi_{xz} = -n\tau k_{\rm B} T\, \frac{\partial V_x}{\partial z} \tag{8.123}$$

from which we read off the viscosity,

$$\eta = n k_{\rm B} T \tau = \frac{\pi}{8} nm\ell\bar{v} \quad . \tag{8.124}$$

Note that $\eta(T) \propto T^{1/2}$.

How well do these predictions hold up? In fig. 8.5, we plot data for the thermal conductivity of argon and the shear viscosity of helium. Both show a clear sublinear behavior as a function of temperature, but the slope $d\log\kappa/d\log T$ is approximately 0.65 and $d\log\eta/d\log T$ is approximately 0.63. Clearly the simple model is not even getting the functional dependence on T right, let alone its coefficient. Still, our crude theory is at least qualitatively correct.

Why do both $\kappa(T)$ as well as $\eta(T)$ decrease at low temperatures? The reason is that the heat current which flows in response to $\boldsymbol{\nabla} T$ as well as the momentum current which flows in response to $\partial V_x/\partial z$

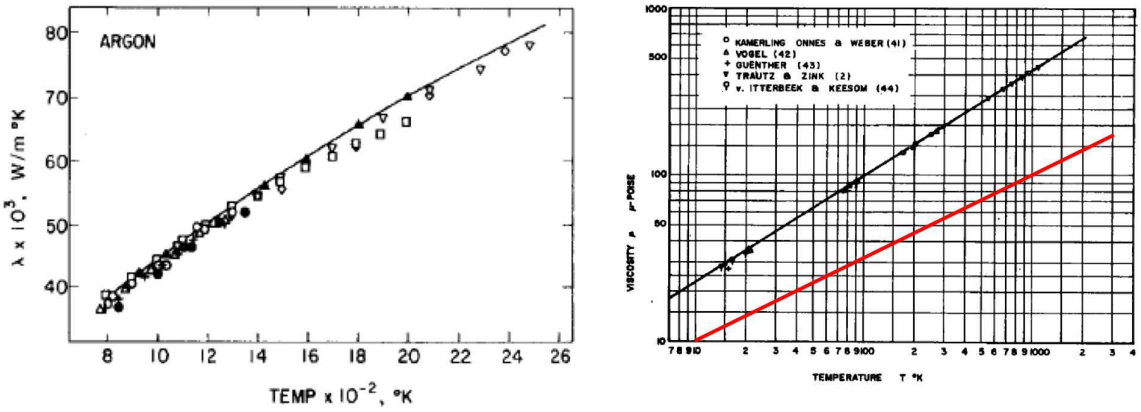


Figure 8.5: Left: thermal conductivity (λ in figure) of Ar between $T = 800\,\mathrm{K}$ and $T = 2600\,\mathrm{K}$. The best fit to a single power law $\lambda = aT^b$ results in $b = 0.651$. Source: G. S. Springer and E. W. Wingeier, *J. Chem Phys.* **59**, 2747 (1973). Right: log-log plot of shear viscosity (μ in figure) of He between $T \approx 15\,\mathrm{K}$ and $T \approx 1000\,\mathrm{K}$. The red line has slope $\frac{1}{2}$. The slope of the data is approximately 0.633. Source: J. Kestin and W. Leidenfrost, *Physica* **25**, 537 (1959).

are due to the presence of collisions, which result in momentum and energy transfer between particles. This is true even when total energy and momentum are conserved, which they are not in the relaxation time approximation. Intuitively, we might think that the viscosity should *increase* as the temperature is lowered, since common experience tells us that fluids 'gum up' as they get colder – think of honey as an extreme example. But of course honey is nothing like an ideal gas, and the physics behind the crystallization or glass transition which occurs in real fluids when they get sufficiently cold is completely absent from our approach. In our calculation, viscosity results from collisions, and with no collisions there is no momentum transfer among the gas molecules and hence no local viscosity. If, for example, the gas particles were to simply pass through each other, as though they were ghosts, all the dissipation would occur due to processes taking place outside the system. More properly, in the dilute-gas hydrodynamic regime, collisions redistribute momentum locally and permit a Navier–Stokes description. As the collision rate tends to zero, the hydrodynamic description fails and transport becomes ballistic, so assigning a local viscosity is no longer appropriate[11].

8.5.5 Oscillating external force

Suppose a uniform oscillating external force $\boldsymbol{F}_{\rm ext}(t) = \boldsymbol{F}\, e^{-i\omega t}$ is applied. For a system of charged particles, this force would arise from an external electric field $\boldsymbol{F}_{\rm ext} = q\boldsymbol{E}\, e^{-i\omega t}$, where q is the charge of

[11]To put it another way, momentum transport occurs because moving particles carry momentum between regions of different flow velocity. Collisions establish the local equilibrium needed for a viscosity coefficient to be defined. At fixed temperature and density, the kinetic result $\eta = nk_{\rm B}T\tau = p\tau$ increases with the collision time while the hydrodynamic approximation remains valid. In the collisionless limit transport becomes ballistic and a local Navier–Stokes viscosity is no longer an appropriate description.

each particle. We'll assume $\boldsymbol{\nabla} T = 0$. The Boltzmann equation is then written

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{F} e^{-i\omega t} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = -\frac{f - f^0}{\tau} \quad . \tag{8.125}$$

We again write $f = f^0 + \delta f$, and we assume δf is spatially constant. Thus,

$$\frac{\partial\, \delta f}{\partial t} + \boldsymbol{F} e^{-i\omega t} \cdot \boldsymbol{v}\, \frac{\partial f^0}{\partial \varepsilon} = -\frac{\delta f}{\tau} \quad . \tag{8.126}$$

If we assume $\delta f(t) = \delta f(\omega)\, e^{-i\omega t}$ then the above differential equation is converted to an algebraic equation, with solution

$$\delta f(t) = -\frac{\tau\, e^{-i\omega t}}{1 - i\omega\tau}\, \frac{\partial f^0}{\partial \varepsilon}\, \boldsymbol{F} \cdot \boldsymbol{v} \quad . \tag{8.127}$$

We now compute the particle current:

$$\begin{aligned} j_\alpha(\boldsymbol{r}, t) = \int d^3p\, v_\alpha\, \delta f &= \frac{\tau\, e^{-i\omega t}}{1 - i\omega\tau} \cdot \frac{F_\beta}{k_{\rm B} T} \int d^3p\, f^0(\boldsymbol{p})\, v_\alpha\, v_\beta \\ &= \frac{\tau\, e^{-i\omega t}}{1 - i\omega\tau} \cdot \frac{n F_\alpha}{3 k_{\rm B} T} \int d^3v\, P(\boldsymbol{v})\, \boldsymbol{v}^2 = \frac{n\tau}{m} \cdot \frac{F_\alpha\, e^{-i\omega t}}{1 - i\omega\tau} \quad . \end{aligned} \tag{8.128}$$

If the particles are electrons, with charge $q = -e$, then the electrical current is $(-e)$ times the particle current. We then obtain

$$j_\alpha^{(\rm elec)}(t) = \frac{n e^2 \tau}{m} \cdot \frac{E_\alpha\, e^{-i\omega t}}{1 - i\omega\tau} \equiv \sigma_{\alpha\beta}(\omega)\, E_\beta\, e^{-i\omega t} \quad , \tag{8.129}$$

where

$$\sigma_{\alpha\beta}(\omega) = \frac{n e^2 \tau}{m} \cdot \frac{1}{1 - i\omega\tau}\, \delta_{\alpha\beta} \tag{8.130}$$

is the frequency-dependent electrical conductivity tensor. Of course for fermions such as electrons, we should be using the Fermi distribution in place of the Maxwell-Boltzmann distribution for $f^0(\boldsymbol{p})$. This affects the relation between n and μ only, and the final result for the conductivity tensor $\sigma_{\alpha\beta}(\omega)$ is unchanged.

8.5.6 Quick and dirty calculation of transport coefficients

Suppose we have some averaged intensive quantity ϕ which is spatially dependent through $T(\boldsymbol{r})$ or $\mu(\boldsymbol{r})$ or $\boldsymbol{V}(\boldsymbol{r})$. For simplicity we will write $\phi = \phi(z)$. We wish to compute the current of ϕ across some surface whose equation is $dz = 0$. If the mean free path is ℓ, then the value of ϕ for particles crossing this surface in the $+\hat{\boldsymbol{z}}$ direction is $\phi(z - \ell\cos\theta)$, where θ is the angle the particle's velocity makes with respect to $\hat{\boldsymbol{z}}$, *i.e.* $\cos\theta = v_z/v > 0$. For particles moving in the $-\hat{\boldsymbol{z}}$ direction, we again have $\phi = \phi(z - \ell\cos\theta)$, but with $\cos\theta = v_z/v < 0$. The current of ϕ through this surface is then

$$\boldsymbol{j}_\phi = n\hat{\boldsymbol{z}} \int d^3v\, P(\boldsymbol{v})\, v_z\, \phi(z - \ell\cos\theta) = -n\ell\, \frac{\partial \phi}{\partial z}\, \hat{\boldsymbol{z}} \int d^3v\, P(\boldsymbol{v})\, \frac{v_z^2}{v} = -\tfrac{1}{3} n \bar{v} \ell\, \frac{\partial \phi}{\partial z}\, \hat{\boldsymbol{z}} \quad , \tag{8.131}$$

where $\cos\theta = v_z/v$ and $\bar{v} = (8k_{\rm B}T/\pi m)^{1/2}$ is the average particle speed. If the z-dependence of ϕ comes through the dependence of ϕ on the local temperature T, then we have

$$\boldsymbol{j}_\phi = -\tfrac{1}{3}\, n\ell\bar{v}\, \frac{\partial\phi}{\partial T}\, \boldsymbol{\nabla} T \equiv -K\, \boldsymbol{\nabla} T \quad , \tag{8.132}$$

where

$$K = \tfrac{1}{3} n\ell\bar{v}\, \frac{\partial\phi}{\partial T} \tag{8.133}$$

is the transport coefficient. With $\phi = \langle\varepsilon\rangle$ we have $\partial\phi/\partial T = c_V$, where c_V is the heat capacity per particle at constant volume. We then find $\boldsymbol{j}_q = -\kappa\, \boldsymbol{\nabla} T$ with thermal conductivity $\kappa = \frac{1}{3} n\ell\bar{v}\, c_V$. Our Boltzmann equation calculation (eqn. 8.116) yielded a similar result, but involving c_p rather than c_V, and with a different prefactor.

We can make a similar argument for the viscosity. In this case $\phi = \langle p_x\rangle$ is spatially varying through its dependence on the flow velocity $\boldsymbol{V}(\boldsymbol{r})$. Clearly $\partial\phi/\partial V_x = m$, hence

$$j^z_{p_x} = \Pi_{xz} = -\tfrac{1}{3} nm\ell\bar{v}\, \frac{\partial V_x}{\partial z} \quad , \tag{8.134}$$

from which we identify the viscosity, $\eta = \frac{1}{3} nm\ell\bar{v}$. Once again, this agrees in its functional dependences with the Boltzmann equation calculation in the relaxation time approximation. Only the coefficients differ.

8.5.7 Thermal diffusivity, kinematic viscosity, and Prandtl number

Suppose, under conditions of constant pressure, we add heat q per unit volume to an ideal gas. We know from thermodynamics that its temperature will then increase by an amount $\Delta T = q/nc_p$. If a heat current $\boldsymbol{j}_q$ flows, then the continuity equation for energy flow requires[12]

$$nc_p\, \frac{\partial T}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}_q = 0 \quad . \tag{8.135}$$

Using $\boldsymbol{j}_q = -\kappa\boldsymbol{\nabla} T$, and linearizing about a uniform fluid at rest with constant coefficients, we obtain a diffusion equation for temperature,

$$\frac{\partial T}{\partial t} = \frac{\kappa}{nc_p}\, \nabla^2 T \quad . \tag{8.136}$$

The combination $a \equiv \kappa/nc_p$ is known as the *thermal diffusivity*. Our Boltzmann equation calculation in the relaxation time approximation yielded the result $\kappa = nk_{\rm B}T\tau c_p/m$. Thus, we find $a = k_{\rm B}T\tau/m$ via this method. Note that the dimensions of a are the same as for any diffusion constant D, namely $[a] = L^2/T$.

Another quantity with dimensions of L^2/T is the *kinematic viscosity*, $\nu = \eta/\rho$, where $\rho = nm$ is the mass density. We found $\eta = nk_{\rm B}T\tau$ from the relaxation time approximation calculation, hence $\nu = k_{\rm B}T\tau/m$. The ratio ν/a, called the *Prandtl number*, $\mathsf{Pr} = \eta c_p/m\kappa$, is dimensionless. According to our calculations, $\mathsf{Pr} = 1$. According to table 8.1, most monatomic gases have $\mathsf{Pr} \approx \frac{2}{3}$.

[12] This equation is a linearization of $nc_p D_t T = \boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla} T)$, where $D_t = \partial_t + \boldsymbol{V}\cdot\boldsymbol{\nabla}$ by linearizing at constant p for a normal fluid at rest.

Gas	η (μPa · s)	κ (mW/m · K)	$c_p/k_{\rm B}$	Pr
He	19.5	149	2.50	0.682
Ar	22.3	17.4	2.50	0.666
Xe	22.7	5.46	2.50	0.659
H_2	8.67	179	3.47	0.693
N_2	17.6	25.5	3.53	0.721
O_2	20.3	26.0	3.50	0.711
CH_4	11.2	33.5	4.29	0.74
CO_2	14.8	18.1	4.47	0.71
NH_3	10.1	24.6	4.50	0.90

Table 8.1: Viscosities, thermal conductivities, and Prandtl numbers for some common gases at $T = 293\,{\rm K}$ and $p = 1\,{\rm atm}$. (Source: Table 1.1 of Smith and Jensen, with data for additional polyatomic gases added.)

8.6 Diffusion and the Lorentz model

8.6.1 Failure of the relaxation time approximation

As notes above, the relaxation time approximation need not conserve any of the collisional invariants. It is therefore unsuitable for describing hydrodynamic phenomena such as diffusion. To see this, let $f(\boldsymbol{r}, \boldsymbol{v}, t)$ be the distribution function, here written in terms of $(\boldsymbol{r}, \boldsymbol{v}, t)$ rather than $(\boldsymbol{r}, \boldsymbol{p}, t)$ as before[13]. In the absence of external forces, the Boltzmann equation in the relaxation time approximation is

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} = -\frac{f - f^0}{\tau} \quad . \tag{8.137}$$

We can solve this equation by first defining $\delta f(\boldsymbol{r}, \boldsymbol{v}, t) \equiv f(\boldsymbol{r}, \boldsymbol{v}, t) - f^0(\boldsymbol{v})$, and then taking the Laplace transform in time and the Fourier transform in space,

$$\delta \hat{f}(\boldsymbol{k}, \boldsymbol{v}, s) = \int\limits_0^\infty dt\, e^{-st} \int d^3r\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, \delta f(\boldsymbol{r}, \boldsymbol{v}, t) \quad , \tag{8.138}$$

resulting in

$$\left(s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}\right) \delta \hat{f}(\boldsymbol{k}, \boldsymbol{v}, s) = \delta f(\boldsymbol{k}, \boldsymbol{v}, t=0) \quad \Rightarrow \quad \delta \hat{f}(\boldsymbol{k}, \boldsymbol{v}, s) = \frac{\delta f(\boldsymbol{k}, \boldsymbol{v}, t=0)}{s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}} \quad . \tag{8.139}$$

[13]The difference is trivial, since $\boldsymbol{p} = m\boldsymbol{v}$. Thus $f_v(\boldsymbol{r}, \boldsymbol{v}, t) = m^3 f_p(\boldsymbol{r}, \boldsymbol{p}, t)$ when the Jacobian factor is included.

Taking the inverse transforms,

$$\begin{aligned}\delta f(\boldsymbol{r},\boldsymbol{v},t) &= \int\frac{d^dk}{(2\pi)^d}\int\limits_{\mathcal{C}}\frac{ds}{2\pi i}\,\delta\hat f(\boldsymbol{k},\boldsymbol{v},s)\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,e^{st}\\ &= \int\frac{d^dk}{(2\pi)^d}\,\delta f(\boldsymbol{k},\boldsymbol{v},t=0)\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\int\limits_{\mathcal{C}}\frac{ds}{2\pi i}\,\frac{e^{st}}{s+i\boldsymbol{v}\cdot\boldsymbol{k}+\tau^{-1}}\\ &= \int\frac{d^dk}{(2\pi)^d}\,\delta f(\boldsymbol{k},\boldsymbol{v},t=0)\,e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{v}t)}\,e^{-t/\tau} = \delta f(\boldsymbol{r}-\boldsymbol{v}t,\boldsymbol{v},0)\,e^{-t/\tau}\quad,\end{aligned}\tag{8.140}$$

where $\mathcal{C}$ is an integration contour from $c-i\infty$ to $c+i\infty$ where c is chosen so that $\mathcal{C}$ lies to the right of all singularities of the integrand. Thus, choosing an initial distribution that is localized at $\boldsymbol{r}=0$ and $\boldsymbol{v}=\boldsymbol{v}_0$, we find it evolves according to

$$\delta f(\boldsymbol{r},\boldsymbol{v},0) = N\,\delta(\boldsymbol{r})\,\delta(\boldsymbol{v}-\boldsymbol{v}_0)\quad\Rightarrow\quad\delta f(\boldsymbol{r},\boldsymbol{v},t) = N\,\delta(\boldsymbol{r}-\boldsymbol{v}_0t)\,\delta(\boldsymbol{v}-\boldsymbol{v}_0)\,e^{-t/\tau}\quad.\tag{8.141}$$

This result is profoundly unphysical – it says that particle number is not conserved. Not only that, but the spatial distribution remains instantaneously localized about $\boldsymbol{r}=\boldsymbol{v}t$, whereas we expect that in a model of random impurity elastic scattering the particle velocity distribution should become isotropic.

8.6.2 Modified Boltzmann equation and its solution

To remedy this unphysical aspect, consider the modified Boltzmann equation,

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\frac{\partial f}{\partial\boldsymbol{r}} = \frac{1}{\tau}\left[-f+\int\frac{d\hat{\boldsymbol{v}}}{4\pi}\,f\right]\equiv\frac{1}{\tau}\big(\mathbb{P}-1\big)f\quad,\tag{8.142}$$

where $\mathbb{P}$ is a projector onto a space of isotropic functions of $\boldsymbol{v}$: $\mathbb{P}F=\int\frac{d\hat{\boldsymbol{v}}}{4\pi}\,F(\boldsymbol{v})=\int\frac{d\hat{\boldsymbol{v}}}{4\pi}\,F(v\,\hat{\boldsymbol{v}})$ for any function $F(\boldsymbol{v})$. Note that $\mathbb{P}F$ is a function of the speed $v=|\boldsymbol{v}|$. Since the equilibrium distribution $f^0(\boldsymbol{v})=f^0(v)$ is the Maxwell distribution, it is isotropic in velocity space, *i.e.* $f^0(\boldsymbol{v})=f^0(v)$, we have that $(\mathbb{P}-1)f^0(\boldsymbol{v})=0$, and thus the above modified Boltzmann equation holds for f as well as for δf. Note that the number density $n(\boldsymbol{r},t)=\int d^3v\,f(\boldsymbol{r},\boldsymbol{v},t)$ and number current density $\boldsymbol{j}(\boldsymbol{r},t)=\int d^3v\,f(\boldsymbol{r},\boldsymbol{v},t)\,\boldsymbol{v}$ satisfy the continuity equation

$$\frac{\partial n(\boldsymbol{r},t)}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{j}(\boldsymbol{r},t)=0\quad,\tag{8.143}$$

since integrating over $\hat{\boldsymbol{v}}$ annihilates the RHS of eqn. 8.142 for all $(\boldsymbol{r},v)$. Thus, total particle number is conserved, which is not the case in the naïve relaxation time approximation.

The model in eqn. 8.142 is known as the *Lorentz model*[14]. To solve it, we again begin with the double Laplace-Fourier transform,

$$\hat f(\boldsymbol{k},\boldsymbol{v},s)=\int\limits_0^\infty dt\,e^{-st}\int d^3r\,e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,f(\boldsymbol{r},\boldsymbol{v},t)\quad.\tag{8.144}$$

[14]See the excellent discussion in the book by Krapivsky, Redner, and Ben-Naim, cited in §6.1.

Applying this transform to eqn. 8.142, we obtain

$$\left(s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}\right)\hat{f}(\boldsymbol{k},\boldsymbol{v},s) = \tau^{-1}\,\mathbb{P}\hat{f}(\boldsymbol{k},\boldsymbol{v},s) + f(\boldsymbol{k},\boldsymbol{v},t=0) \quad . \tag{8.145}$$

We now solve for $\mathbb{P}\hat{f}(\boldsymbol{k},\boldsymbol{v},s)$:

$$\hat{f}(\boldsymbol{k},\boldsymbol{v},s) = \frac{\tau^{-1}}{s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}}\,\mathbb{P}\hat{f}(\boldsymbol{k},\boldsymbol{v},s) + \frac{f(\boldsymbol{k},\boldsymbol{v},t=0)}{s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}} \quad , \tag{8.146}$$

which entails

$$\mathbb{P}\hat{f}(\boldsymbol{k},\boldsymbol{v},s) = \left[\int\frac{d\hat{\boldsymbol{v}}}{4\pi}\,\frac{\tau^{-1}}{s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}}\right]\mathbb{P}\hat{f}(\boldsymbol{k},\boldsymbol{v},\boldsymbol{s}) + \int\frac{d\hat{\boldsymbol{v}}}{4\pi}\,\frac{f(\boldsymbol{k},\boldsymbol{v},t=0)}{s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}} \quad . \tag{8.147}$$

Now we have

$$\int\frac{d\hat{\boldsymbol{v}}}{4\pi}\,\frac{\tau^{-1}}{s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}} = \frac{1}{2}\int_{-1}^{1}dx\,\frac{\tau^{-1}}{s + ivkx + \tau^{-1}} = \frac{1}{vk\tau}\tan^{-1}\!\left(\frac{vk\tau}{1+\tau s}\right) \quad . \tag{8.148}$$

Thus,

$$\mathbb{P}\hat{f}(\boldsymbol{k},\boldsymbol{v},s) = \left[1 - \frac{1}{vk\tau}\tan^{-1}\!\left(\frac{vk\tau}{1+\tau s}\right)\right]^{-1}\int\frac{d\hat{\boldsymbol{v}}}{4\pi}\,\frac{f(\boldsymbol{k},\boldsymbol{v},t=0)}{s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}} \quad . \tag{8.149}$$

We now have the solution to Lorentz's modified Boltzmann equation:

$$\hat{f}(\boldsymbol{k},\boldsymbol{v},s) = \frac{\tau^{-1}}{s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}}\left[1 - \frac{1}{vk\tau}\tan^{-1}\!\left(\frac{vk\tau}{1+\tau s}\right)\right]^{-1}\int\frac{d\hat{\boldsymbol{v}}}{4\pi}\,\frac{f(\boldsymbol{k},\boldsymbol{v},t=0)}{s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}} + \frac{f(\boldsymbol{k},\boldsymbol{v},t=0)}{s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}} \quad . \tag{8.150}$$

Thus,

$$f(\boldsymbol{r},\boldsymbol{v},t) = \int\frac{d^3k}{(2\pi)^3}\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\int_{\mathcal{C}}\frac{ds}{2\pi i}\,e^{st}\left\{\frac{\tau^{-1}}{s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}}\left[1 - \frac{1}{vk\tau}\tan^{-1}\!\left(\frac{vk\tau}{1+\tau s}\right)\right]^{-1}\int\frac{d\hat{\boldsymbol{v}}}{4\pi}\,\frac{f(\boldsymbol{k},\boldsymbol{v},t=0)}{s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}}\right\}$$
$$+ f(\boldsymbol{r} - \boldsymbol{v}t,\boldsymbol{v},0)\,e^{-t/\tau} \quad . \tag{8.151}$$

Note that the last term $f(\boldsymbol{r} - \boldsymbol{v}t,\boldsymbol{v},0)\,\exp(-t/\tau)$ vanishes as $t \to \infty$. However, we have already seen that the full expression must satisfy the continuity equation for $n(\boldsymbol{r},t)$. We will now show that in the long time limit, when the exponentially decaying transient may be neglected, the remaining result is an expanding cloud describing isotropic particle diffusion.

Let us again consider an initial distribution which is perfectly localized in both $\boldsymbol{r}$ and $\boldsymbol{v}$:

$$f(\boldsymbol{r},\boldsymbol{v},t=0) = N\,\delta(\boldsymbol{r})\,\delta(\boldsymbol{v} - \boldsymbol{v}_0) \quad . \tag{8.152}$$

For these initial conditions, we find

$$\int\frac{d\hat{\boldsymbol{v}}}{4\pi}\,\frac{f(\boldsymbol{k},\boldsymbol{v},t=0)}{s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}} = \frac{1}{s + i\boldsymbol{v}_0\cdot\boldsymbol{k} + \tau^{-1}}\cdot\frac{N\,\delta(v - v_0)}{4\pi v_0^2} \quad . \tag{8.153}$$

We are interested in the long time limit of $f(\boldsymbol{r}, v, t)$, where $t \gg \tau$. Long times are dominated in the Laplace transform by $s \sim t^{-1}$. We also assume the distribution in space becomes smooth, as we shall show, and expand in the regime where $s\tau \ll 1$ and $vk\tau \ll 1$. We then have

$$1 - \frac{1}{vk\tau} \tan^{-1}\left(\frac{vk\tau}{1+\tau s}\right) = s\tau + \tfrac{1}{3}k^2v^2\tau^2 + \dots \quad , \tag{8.154}$$

and therefore

$$\begin{aligned}\hat{f}(\boldsymbol{k}, \boldsymbol{v}, s) &= \frac{\tau^{-1}}{s + i\boldsymbol{v}\cdot\boldsymbol{k} + \tau^{-1}} \cdot \frac{\tau^{-1}}{s + i\boldsymbol{v}_0\cdot\boldsymbol{k} + \tau^{-1}} \cdot \frac{1}{s + \frac{1}{3}v_0^2\,k^2\,\tau + \dots} \cdot \frac{N\,\delta(v - v_0)}{4\pi v_0^2} + \frac{N\,\delta(\boldsymbol{v} - \boldsymbol{v}_0)}{s + i\boldsymbol{v}_0\cdot\boldsymbol{k} + \tau^{-1}} \\ &\approx \frac{1}{s + \frac{1}{3}v_0^2\,k^2\,\tau} \cdot \frac{N\,\delta(v - v_0)}{4\pi v_0^2} + \frac{N\,\delta(\boldsymbol{v} - \boldsymbol{v}_0)}{s + i\boldsymbol{v}_0\cdot\boldsymbol{k} + \tau^{-1}} \quad ,\end{aligned} \tag{8.155}$$

since τ^{-1} is dominant over s and $i\boldsymbol{v}\cdot\boldsymbol{k}$ in the denominators of the first two multiplicative factors on the RHS of the top equation. We then have

$$\hat{f}(\boldsymbol{k}, \boldsymbol{v}, s) \approx \frac{1}{s + \frac{1}{3}v_0^2\,k^2\,\tau} \cdot \frac{N\,\delta(v - v_0)}{4\pi v_0^2} + \frac{N\,\delta(\boldsymbol{v} - \boldsymbol{v}_0)}{s + i\boldsymbol{v}_0\cdot\boldsymbol{k} + \tau^{-1}} \quad . \tag{8.156}$$

Performing the inverse Laplace and Fourier transforms, and dropping the transient term for $t \gg \tau$, we obtain our final result,

$$f(\boldsymbol{r}, \boldsymbol{v}, t \gg \tau) = (4\pi D t)^{-3/2}\, e^{-r^2/4Dt} \cdot \frac{N\,\delta(v - v_0)}{4\pi v_0^2} \quad , \tag{8.157}$$

where the *diffusion constant* is $D = \frac{1}{3}v_0^2\,\tau$. The units are $[D] = L^2/T$. Integrating over velocities, we have the density

$$n(\boldsymbol{r}, t \gg \tau) = \int d^3v\; f(\boldsymbol{r}, \boldsymbol{v}, t \gg \tau) = N\,(4\pi D t)^{-3/2}\, e^{-r^2/4Dt} \quad . \tag{8.158}$$

Note that $\int d^3r\, n(\boldsymbol{r}, t \gg \tau) = N$ at all times in this limit. In addition to particle number being conserved, we see that the late time distribution $f(\boldsymbol{r}, \boldsymbol{v}, t \gg \tau)$ is isotropic in both $\boldsymbol{r}$ and $\boldsymbol{v}$ to leading order in the diffusive scaling limit.

8.7 Linearized Boltzmann Equation

8.7.1 Linearizing the collision integral

We now return to the classical Boltzmann equation and consider a more formal treatment of the collision term in the linear approximation. We will assume time-reversal and parity symmetry, in which case

$$\left(\frac{df}{dt}\right)_{\rm coll} = \int d^3p_1 \int d^3p' \int d^3p_1'\; w(\boldsymbol{p}', \boldsymbol{p}_1' \,|\, \boldsymbol{p}, \boldsymbol{p}_1)\,\Big\{f(\boldsymbol{p}')\,f(\boldsymbol{p}_1') - f(\boldsymbol{p})\,f(\boldsymbol{p}_1)\Big\} \quad . \tag{8.159}$$

The collision integral is nonlinear in the distribution f. We *linearize* by writing

$$f(\boldsymbol{p}) = f^0(\boldsymbol{p}) + f^0(\boldsymbol{p})\,\psi(\boldsymbol{p}) \quad , \tag{8.160}$$

where we assume $\psi(\boldsymbol{p})$ is small. We then have, to first order in ψ,

$$\left(\frac{df}{dt}\right)_{\rm coll} = f^0(\boldsymbol{p})\,\hat{L}\psi + \mathcal{O}(\psi^2) \quad , \tag{8.161}$$

where the action of the *linearized collision operator* is given by

$$\begin{aligned}\hat{L}\psi &= \int d^3p_1 \int d^3p' \int d^3p_1'\; w(\boldsymbol{p}', \boldsymbol{p}_1' \,|\, \boldsymbol{p}, \boldsymbol{p}_1)\; f^0(\boldsymbol{p}_1)\left\{\psi(\boldsymbol{p}') + \psi(\boldsymbol{p}_1') - \psi(\boldsymbol{p}) - \psi(\boldsymbol{p}_1)\right\} \\ &= \int d^3p_1 \int d\Omega\, |\boldsymbol{v} - \boldsymbol{v}_1| \,\frac{d\sigma}{d\Omega}\, f^0(\boldsymbol{p}_1)\left\{\psi(\boldsymbol{p}') + \psi(\boldsymbol{p}_1') - \psi(\boldsymbol{p}) - \psi(\boldsymbol{p}_1)\right\} \quad ,\end{aligned} \tag{8.162}$$

where we have invoked eqn. 8.61 to write the RHS in terms of the differential scattering cross section. In deriving the above result, we have made use of the detailed balance relation,

$$f^0(\boldsymbol{p})\, f^0(\boldsymbol{p}_1) = f^0(\boldsymbol{p}')\, f^0(\boldsymbol{p}_1') \quad . \tag{8.163}$$

We have also suppressed the $\boldsymbol{r}$ dependence in writing $f(\boldsymbol{p})$, $f^0(\boldsymbol{p})$, and $\psi(\boldsymbol{p})$.

From eqn. 8.98, we then have the linearized equation

$$\left(\hat{L} - \frac{\partial}{\partial t}\right)\psi = Y, \tag{8.164}$$

where, for point particles,

$$Y = \frac{1}{k_{\rm B}T}\left\{\frac{\varepsilon(\boldsymbol{p}) - c_p T}{T}\,\boldsymbol{v}\cdot\boldsymbol{\nabla}T + m\,v_\alpha v_\beta\,\mathcal{Q}_{\alpha\beta} - \frac{k_{\rm B}\,\varepsilon(\boldsymbol{p})}{c_V}\,\boldsymbol{\nabla}\cdot\boldsymbol{V}\right\} \quad . \tag{8.165}$$

Eqn. 8.164 is an inhomogeneous linear equation, which can be solved by inverting the operator $\hat{L} - \frac{\partial}{\partial t}$.

8.7.2 Linear algebraic properties of $\hat{L}$

Although $\hat{L}$ is an integral operator, it shares many properties with other linear operators with which you are familiar, such as matrices and differential operators. We can define an *inner product*[15],

$$\langle\, \psi_1 \,|\, \psi_2 \,\rangle \equiv \int d^3p\, f^0(\boldsymbol{p})\, \psi_1^*(\boldsymbol{p})\, \psi_2(\boldsymbol{p}) \quad . \tag{8.166}$$

Note that this is not the usual Hilbert space inner product from quantum mechanics, since the factor $f^0(\boldsymbol{p})$ is included in the metric. This is necessary in order that $\hat{L}$ be *self-adjoint*: $\langle\, \psi_1 \,|\, \hat{L}\psi_2 \,\rangle = \langle\, \hat{L}\psi_1 \,|\, \psi_2 \,\rangle$.

[15] The requirements of an inner product $\langle\, f \,|\, g \,\rangle$ are conjugate symmetry, linearity in g, conjugate linearity in f, and positive definiteness.

We can now define the spectrum of *normalized eigenfunctions* of $\hat{L}$, which we write as $\phi_n(\boldsymbol{p})$. The eigenfunctions satisfy the eigenvalue equation,

$$\hat{L}\,\phi_n = -\lambda_n\,\phi_n \quad , \tag{8.167}$$

and may be chosen to be orthonormal, *i.e.* $\langle\,\phi_m\,|\,\phi_n\,\rangle = \delta_{mn}$. Of course, in order to obtain the eigenfunctions ϕ_n we must have detailed knowledge of the function $w(\boldsymbol{p}',\boldsymbol{p}_1'\,|\,\boldsymbol{p},\boldsymbol{p}_1)$.

Recall that there are five collisional invariants, which are the particle number, the three components of the total particle momentum, and the particle energy. To each collisional invariant, there is an associated eigenfunction ϕ_n with eigenvalue $\lambda_n = 0$. One can check that these normalized eigenfunctions are

$$\phi_n(\boldsymbol{p}) = \frac{1}{\sqrt{n}} \qquad , \qquad \phi_{p_\alpha}(\boldsymbol{p}) = \frac{p_\alpha}{\sqrt{nmk_{\rm B}T}} \qquad , \qquad \phi_\varepsilon(\boldsymbol{p}) = \sqrt{\frac{2}{3n}}\left(\frac{\varepsilon(\boldsymbol{p})}{k_{\rm B}T} - \frac{3}{2}\right) \quad . \tag{8.168}$$

If there are no temperature, chemical potential, or bulk velocity gradients, and there are no external forces, then $Y = 0$ and the only changes to the distribution are from collisions. The linearized Boltzmann equation becomes

$$\frac{\partial\psi}{\partial t} = \hat{L}\psi \quad . \tag{8.169}$$

We can therefore write the most general solution in the form[16]

$$\psi(\boldsymbol{p},t) = {\sum_n}' C_n\,\phi_n(\boldsymbol{p})\,e^{-\lambda_n t} \quad , \tag{8.170}$$

where the prime on the sum reminds us that collisional invariants are to be excluded. All the eigenvalues λ_n, aside from the five zero eigenvalues for the collisional invariants, must be positive. Any negative eigenvalue would cause $\psi(\boldsymbol{p},t)$ to increase without bound, and an initial nonequilibrium distribution would not relax to the equilibrium $f^0(\boldsymbol{p})$, which we regard as unphysical.

Recall also the particle, energy, and thermal currents,

$$\begin{aligned}
\boldsymbol{j} &= \int d^3p\,\boldsymbol{v}\,f(\boldsymbol{p}) = \int d^3p\,f^0(\boldsymbol{p})\,\boldsymbol{v}\,\psi(\boldsymbol{p}) = \langle\,\boldsymbol{v}\,|\,\psi\,\rangle \\
\boldsymbol{j}_\varepsilon &= \int d^3p\,\boldsymbol{v}\,\varepsilon\,f(\boldsymbol{p}) = \int d^3p\,f^0(\boldsymbol{p})\,\boldsymbol{v}\,\varepsilon\,\psi(\boldsymbol{p}) = \langle\,\boldsymbol{v}\,\varepsilon\,|\,\psi\,\rangle \\
\boldsymbol{j}_{\rm th} &= \int d^3p\,\boldsymbol{v}\,(\varepsilon-\mu)\,f(\boldsymbol{p}) = \int d^3p\,f^0(\boldsymbol{p})\,\boldsymbol{v}\,(\varepsilon-\mu)\,\psi(\boldsymbol{p}) = \langle\,\boldsymbol{v}\,(\varepsilon-\mu)\,|\,\psi\,\rangle \quad .
\end{aligned} \tag{8.171}$$

Note $\boldsymbol{j}_{\rm th} = \boldsymbol{j}_\varepsilon - \mu\boldsymbol{j}$.

[16]We choose the reference Maxwellian to have the same conserved moments as the initial distribution, so that ψ is orthogonal to the five collisional invariants. We also write the spectral decomposition as a discrete sum; continuous spectral components, when present, are represented by corresponding integrals.

8.7.3 Steady state solution to the linearized Boltzmann equation

Under steady state conditions, there is no time dependence, and the linearized Boltzmann equation takes the form $\hat{L}\psi = Y$. We may expand ψ in the eigenfunctions ϕ_n and write $\psi = \sum_n C_n\,\phi_n$. Applying $\hat{L}$ and taking the inner product with ϕ_j, we have for the nonconserved modes j with $\lambda_j > 0$,

$$C_j = -\frac{1}{\lambda_j}\,\langle\,\phi_j\,|\,Y\,\rangle \quad . \tag{8.172}$$

Thus, the formal solution to the linearized Boltzmann equation is

$$\psi(\boldsymbol{p}) = -{\sum_n}'\,\frac{1}{\lambda_n}\,\langle\,\phi_n\,|\,Y\,\rangle\,\phi_n(\boldsymbol{p}) \quad . \tag{8.173}$$

This solution is applicable provided $|\,Y\,\rangle$ is orthogonal to the five collisional invariants.

Thermal conductivity

For the thermal conductivity, we take $\boldsymbol{\nabla} T = \partial_x T\,\hat{\boldsymbol{x}}$, and

$$Y = \frac{1}{k_{\rm B}T^2}\,\frac{\partial T}{\partial x}\cdot X_\kappa \quad , \tag{8.174}$$

where $X_\kappa \equiv (\varepsilon - c_p T)\,v_x$. Under the conditions of no particle flow ($\boldsymbol{j} = 0$), we have $\boldsymbol{j}_q = -\kappa\,\partial_x T\,\hat{\boldsymbol{x}}$. Then we have

$$\langle\,X_\kappa\,|\,\psi\,\rangle = -\kappa\,\frac{\partial T}{\partial x} \quad . \tag{8.175}$$

Viscosity

For the viscosity, we take

$$Y = \frac{m}{k_{\rm B}T}\,\frac{\partial V_x}{\partial y}\cdot X_\eta \quad , \tag{8.176}$$

with $X_\eta = v_x\,v_y$. We then

$$\Pi_{xy} = \langle\,m\,v_x\,v_y\,|\,\psi\,\rangle = -\eta\,\frac{\partial V_x}{\partial y} \quad . \tag{8.177}$$

Thus,

$$\langle\,X_\eta\,|\,\psi\,\rangle = -\frac{\eta}{m}\,\frac{\partial V_x}{\partial y} \quad . \tag{8.178}$$

8.7.4 Variational approach

Following the treatment in chapter 1 of Smith and Jensen, define $\hat{H} \equiv -\hat{L}$. We have that $\hat{H}$ is a positive semidefinite operator, whose only zero eigenvalues correspond to the collisional invariants. We then

have the Schwarz inequality,

$$\langle\,\psi\,|\,\hat{H}\,|\,\psi\,\rangle\cdot\langle\,\phi\,|\,\hat{H}\,|\,\phi\,\rangle \geq \left|\langle\,\phi\,|\,\hat{H}\,|\,\psi\,\rangle\right|^2 \quad , \tag{8.179}$$

for any two Hilbert space vectors $|\,\psi\,\rangle$ and $|\,\phi\,\rangle$. Consider now the above calculation of the thermal conductivity. We have

$$\hat{H}\psi = -\frac{1}{k_{\rm B}T^2}\frac{\partial T}{\partial x}\,X_\kappa \tag{8.180}$$

and therefore

$$\kappa = \frac{k_{\rm B}T^2}{(\partial T/\partial x)^2}\,\langle\,\psi\,|\,\hat{H}\,|\,\psi\,\rangle \geq \frac{1}{k_{\rm B}T^2}\,\frac{\left|\langle\,\phi\,|\,X_\kappa\,\rangle\right|^2}{\langle\,\phi\,|\,\hat{H}\,|\,\phi\,\rangle} \quad . \tag{8.181}$$

Similarly, for the viscosity, we have

$$\hat{H}\psi = -\frac{m}{k_{\rm B}T}\frac{\partial V_x}{\partial y}\,X_\eta \quad , \tag{8.182}$$

from which we derive

$$\eta = \frac{k_{\rm B}T}{(\partial V_x/\partial y)^2}\,\langle\,\psi\,|\,\hat{H}\,|\,\psi\,\rangle \geq \frac{m^2}{k_{\rm B}T}\,\frac{\left|\langle\,\phi\,|\,X_\eta\,\rangle\right|^2}{\langle\,\phi\,|\,\hat{H}\,|\,\phi\,\rangle} \quad . \tag{8.183}$$

In order to get a good lower bound, we want ϕ in each case to have a good overlap with $X_{\kappa,\eta}$. One approach then is to take $\phi = X_{\kappa,\eta}$, which guarantees that the overlap will be finite (and not zero due to symmetry, for example). We illustrate this method with the viscosity calculation. We have

$$\eta \geq \frac{m^2}{k_{\rm B}T}\,\frac{\langle\,v_xv_y\,|\,v_xv_y\,\rangle^2}{\langle\,v_xv_y\,|\,\hat{H}\,|\,v_xv_y\,\rangle} \quad . \tag{8.184}$$

With $\hat{H} = -\hat{L}$ we have

$$\langle\,\phi\,|\,\hat{H}\,|\,\psi\,\rangle = \int d^3p\,f^0(\boldsymbol{p})\,\phi^*(\boldsymbol{p})\int d^3p_1\int d\Omega\,\frac{d\sigma}{d\Omega}\,|\boldsymbol{v}-\boldsymbol{v}_1|\,f^0(\boldsymbol{p}_1)\left\{\psi(\boldsymbol{p})+\psi(\boldsymbol{p}_1)-\psi(\boldsymbol{p}')-\psi(\boldsymbol{p}_1')\right\} \quad . \tag{8.185}$$

Here the kinematics of the collision guarantee total energy and momentum conservation, so $\boldsymbol{p}'$ and $\boldsymbol{p}_1'$ are determined as in eqn. 8.63.

We have $d\Omega = \sin\chi\,d\chi\,d\varphi$, where χ is the scattering angle depicted in fig. 8.6 and φ is the azimuthal angle of the scattering. The differential scattering cross section is obtained by elementary mechanics and is known to be

$$\frac{d\sigma}{d\Omega} = \left|\frac{d(b^2/2)}{d\cos\chi}\right| \quad , \tag{8.186}$$

where b is the *impact parameter*. The scattering angle is

$$\chi(b,u) = \pi - 2\int\limits_{r_{\rm p}}^{\infty} dr\,\frac{b}{\sqrt{r^4 - b^2r^2 - \frac{2U(r)r^4}{\tilde{m}u^2}}} \quad , \tag{8.187}$$

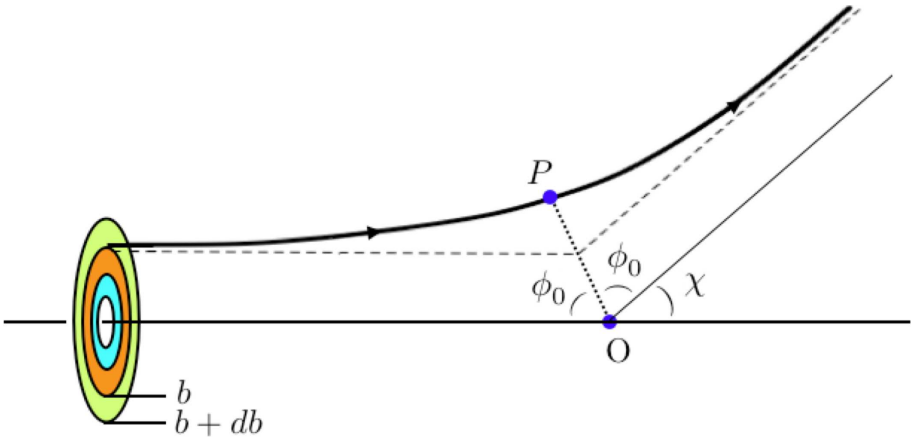


Figure 8.6: Scattering in the CM frame. O is the force center and P is the point of periapsis. The impact parameter is b, and χ is the scattering angle. ϕ_0 is the angle through which the relative coordinate moves between periapsis and infinity.

where $\widetilde{m} = \frac{1}{2}m$ is the reduced mass, and $r_{\rm p}$ is the relative coordinate separation at periapsis, *i.e.* the distance of closest approach, which occurs when $\dot{r} = 0$, *i.e.*

$$\tfrac{1}{2}\widetilde{m}u^2 = \frac{\ell^2}{2\widetilde{m}r_{\rm p}^2} + U(r_{\rm p}) \quad , \tag{8.188}$$

where $\ell = \widetilde{m}ub$ is the relative coordinate angular momentum.

We work in center-of-mass coordinates, so the velocities are

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{V} + \tfrac{1}{2}\boldsymbol{u} \qquad & \boldsymbol{v}' &= \boldsymbol{V} + \tfrac{1}{2}\boldsymbol{u}' \\ \boldsymbol{v}_1 &= \boldsymbol{V} - \tfrac{1}{2}\boldsymbol{u} \qquad & \boldsymbol{v}_1' &= \boldsymbol{V} - \tfrac{1}{2}\boldsymbol{u}' \end{aligned} \quad , \tag{8.189}$$

with $|\boldsymbol{u}| = |\boldsymbol{u}'|$ and $\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}}' = \cos\chi$. Then if $\psi(\boldsymbol{p}) = v_x v_y$, we have

$$\Delta(\psi) \equiv \psi(\boldsymbol{p}) + \psi(\boldsymbol{p}_1) - \psi(\boldsymbol{p}') - \psi(\boldsymbol{p}_1') = \tfrac{1}{2}\big(u_x u_y - u_x' u_y'\big) \quad . \tag{8.190}$$

We may write

$$\boldsymbol{u}' = u\left(\sin\chi\cos\varphi\,\hat{\mathbf{e}}_1 + \sin\chi\sin\varphi\,\hat{\mathbf{e}}_2 + \cos\chi\,\hat{\mathbf{e}}_3\right) \quad , \tag{8.191}$$

where $\hat{\mathbf{e}}_3 = \hat{\boldsymbol{u}}$. With this parameterization, we have

$$\int_0^{2\pi} d\varphi\,\left(u_\alpha u_\beta - u_\alpha' u_\beta'\right) = -\pi\sin^2\chi\,\left(u^2\,\delta_{\alpha\beta} - 3u_\alpha u_\beta\right) \quad . \tag{8.192}$$

Note that we have used here the relation

$$e_{1\alpha}\,e_{1\beta} + e_{2\alpha}\,e_{2\beta} + e_{3\alpha}\,e_{3\beta} = \delta_{\alpha\beta} \quad , \tag{8.193}$$

which holds since the LHS is a projector $\sum_{i=1}^3 |\hat{\mathbf{e}}_i\rangle\langle\hat{\mathbf{e}}_i|$.

It is convenient to define the following integral:

$$R(u) \equiv \int_0^\infty db\, b\, \sin^2\chi(b,u) \quad . \tag{8.194}$$

Since the Jacobian satisfies

$$\left| \det \frac{(\partial \boldsymbol{v}, \partial \boldsymbol{v}_1)}{(\partial \boldsymbol{V}, \partial \boldsymbol{u})} \right| = 1 \quad , \tag{8.195}$$

we have

$$\langle\, v_x v_y \,|\, \hat{H} \,|\, v_x v_y \,\rangle = n^2 \left(\frac{m}{2\pi k_{\rm B} T} \right)^3 \int d^3V \int d^3u\, e^{-m\boldsymbol{V}^2/k_{\rm B}T}\, e^{-m\boldsymbol{u}^2/4k_{\rm B}T} \cdot u \cdot \tfrac{3\pi}{2} u_x u_y \cdot R(u) \cdot v_x v_y \quad . \tag{8.196}$$

This yields

$$\langle\, v_x v_y \,|\, \hat{H} \,|\, v_x v_y \,\rangle = \tfrac{\pi}{40}\, n^2 \left\langle u^5\, R(u) \right\rangle \quad , \tag{8.197}$$

where

$$\left\langle F(u) \right\rangle \equiv \int_0^\infty du\, u^2\, e^{-mu^2/4k_{\rm B}T}\, F(u) \Big/ \int_0^\infty du\, u^2\, e^{-mu^2/4k_{\rm B}T} \quad . \tag{8.198}$$

It is easy to compute the term in the numerator of eqn. 8.184:

$$\langle\, v_x v_y \,|\, v_x v_y \,\rangle = n \left(\frac{m}{2\pi k_{\rm B} T} \right)^{3/2} \int d^3v\, e^{-m\boldsymbol{v}^2/2k_{\rm B}T}\, v_x^2\, v_y^2 = n \left(\frac{k_{\rm B} T}{m} \right)^2 \quad . \tag{8.199}$$

Putting it all together, we find

$$\eta \geq \frac{40\, (k_{\rm B}T)^3}{\pi\, m^2} \Big/ \left\langle u^5\, R(u) \right\rangle \quad . \tag{8.200}$$

The computation for κ is a bit more tedious. One has $\psi(\boldsymbol{p}) = (\varepsilon - c_p T)\, v_x$, in which case

$$\Delta(\psi) = \tfrac{1}{2} m \Big[(\boldsymbol{V} \cdot \boldsymbol{u})\, u_x - (\boldsymbol{V} \cdot \boldsymbol{u}')\, u_x' \Big] \quad . \tag{8.201}$$

Ultimately, one obtains the lower bound

$$\kappa \geq \frac{150\, k_{\rm B}\, (k_{\rm B}T)^3}{\pi\, m^3} \Big/ \left\langle u^5\, R(u) \right\rangle \quad . \tag{8.202}$$

Thus, independent of the potential, taking the ratio of these individual variational estimates yields a Prandtl number of

$$\mathsf{Pr} = \frac{\nu}{a} = \frac{\eta\, c_p}{m\, \kappa} = \frac{2}{3} \quad , \tag{8.203}$$

which is very close to what is observed in dilute monatomic gases (see Tab. 8.1).

While the variational expressions for η and κ are complicated functions of the potential, for hard sphere scattering the calculation is simple, because $b = d\sin\phi_0 = d\cos(\frac{1}{2}\chi)$, where d is the hard sphere diameter. Thus, the impact parameter b is independent of the relative speed u, and one finds $R(u) = \frac{1}{3}d^2$. Then

$$\big\langle u^5\, R(u)\big\rangle = \tfrac{1}{3}d^2\big\langle u^5\big\rangle = \frac{128}{\sqrt{\pi}}\left(\frac{k_{\rm B}T}{m}\right)^{5/2} d^2 \tag{8.204}$$

and one finds

$$\eta \geq \frac{5\,(mk_{\rm B}T)^{1/2}}{16\sqrt{\pi}\,d^2} \quad , \qquad \kappa \geq \frac{75\,k_{\rm B}}{64\sqrt{\pi}\,d^2}\left(\frac{k_{\rm B}T}{m}\right)^{1/2} \quad . \tag{8.205}$$

8.8 The Equations of Hydrodynamics

We now derive the equations governing fluid flow. The equations of mass and momentum balance are

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{V}) = 0 \tag{8.206}$$

$$\frac{\partial(\rho\,V_\alpha)}{\partial t} + \frac{\partial\Pi_{\alpha\beta}}{\partial x^\beta} = 0 \quad , \tag{8.207}$$

where

$$\Pi_{\alpha\beta} = \rho\,V_\alpha V_\beta + p\,\delta_{\alpha\beta} - \overbrace{\left\{\eta\left(\frac{\partial V_\alpha}{\partial x_\beta} + \frac{\partial V_\beta}{\partial x_\alpha} - \tfrac{2}{3}\,\boldsymbol{\nabla}\cdot\boldsymbol{V}\,\delta_{\alpha\beta}\right) + \zeta\,\boldsymbol{\nabla}\cdot\boldsymbol{V}\,\delta_{\alpha\beta}\right\}}^{\tilde\sigma_{\alpha\beta}} \quad . \tag{8.208}$$

Substituting the continuity equation into the momentum balance equation, one arrives at[17]

$$\rho\,\frac{\partial\boldsymbol{V}}{\partial t} + \rho\,(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V} = -\boldsymbol{\nabla}p + \eta\,\nabla^2\boldsymbol{V} + (\zeta + \tfrac{1}{3}\eta)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{V}) \quad , \tag{8.209}$$

which, together with continuity, are known as the *Navier-Stokes equations*. These equations are supplemented by an equation describing the conservation of energy,

$$nT\left(\frac{\partial s}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}s\right) = \tilde\sigma_{\alpha\beta}\,\frac{\partial V_\alpha}{\partial x^\beta} + \boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T) \quad , \tag{8.210}$$

where s is the entropy per particle and $\tilde\sigma_{\alpha\beta}$ is the viscosity stress tensor of eqn. 8.119 (also identified in eqn. 8.208 above). Note that the LHS of eqn. 8.209 is $\rho\,D\boldsymbol{V}/Dt$, where D/Dt is the convective derivative. Multiplying by a differential volume, this gives the mass times the acceleration of a differential local fluid element. The RHS, multiplied by the same differential volume, gives the differential force on this fluid element in a frame instantaneously moving with constant velocity $\boldsymbol{V}$. Thus, this is Newton's Second Law for the fluid.

[17] We assume constant η and ζ.

If the fluid is incompressible, then $\boldsymbol{\nabla}\cdot\boldsymbol{V}=0$, and with the kinematic viscosity $\nu\equiv\eta/\rho$ we have the incompressible Navier-Stokes equations,

$$\frac{\partial\boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}=-\frac{1}{\rho}\,\boldsymbol{\nabla}p+\nu\,\nabla^2\boldsymbol{V}\quad . \tag{8.211}$$

Since $[\eta]=M/LT$ and $[\rho]=M/L^3$, we have $[\nu]=L^2/T$. At $T=20^\circ\mathrm{C}$,

$$\nu_{\mathrm{H_2O}}=10^{-2}\,\mathrm{cm}^2/\mathrm{sec}\qquad,\qquad\nu_{\mathrm{air}}=0.15\,\mathrm{cm}^2/\mathrm{sec}\qquad,\qquad\nu_{\mathrm{glycerine}}=6.8\,\mathrm{cm}^2/\mathrm{sec}\quad . \tag{8.212}$$

8.9 Nonequilibrium Quantum Transport

8.9.1 Boltzmann equation for quantum systems

Almost everything we have derived thus far can be applied, *mutatis mutandis*, to quantum systems. The main difference is that the distribution f^0 corresponding to local equilibrium is no longer of the Maxwell-Boltzmann form, but rather of the Bose-Einstein or Fermi-Dirac form,

$$f^0(\boldsymbol{r},\boldsymbol{k},t)=\left\{\exp\left(\frac{\varepsilon(\boldsymbol{k})-\mu(\boldsymbol{r},t)}{k_{\mathrm{B}}T(\boldsymbol{r},t)}\right)\mp1\right\}^{-1}\quad , \tag{8.213}$$

where the top sign applies to bosons and the bottom sign to fermions[18]. Here we shift to the more common notation for quantum systems in which we write the distribution in terms of the wavevector $\boldsymbol{k}=\boldsymbol{p}/\hbar$ rather than the momentum $\boldsymbol{p}$. The quantum distributions satisfy detailed balance with respect to the quantum collision integral

$$\left(\frac{df}{dt}\right)_{\mathrm{coll}}=\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\int\frac{d^3k_1'}{(2\pi)^3}\,w\left\{f'f_1'\,(1\pm f)\,(1\pm f_1)-f f_1\,(1\pm f')\,(1\pm f_1')\right\} \tag{8.214}$$

where $w=w(\boldsymbol{k},\boldsymbol{k}_1\,|\,\boldsymbol{k}',\boldsymbol{k}_1')$, $f=f(\boldsymbol{k})$, $f_1=f(\boldsymbol{k}_1)$, $f'=f(\boldsymbol{k}')$, and $f_1'=f(\boldsymbol{k}_1')$, and where we have assumed time-reversal and parity symmetry. Detailed balance requires

$$\frac{f}{1\pm f}\cdot\frac{f_1}{1\pm f_1}=\frac{f'}{1\pm f'}\cdot\frac{f_1'}{1\pm f_1'}\quad , \tag{8.215}$$

where $f=f^0$ is the equilibrium distribution. One can check that

$$f=\frac{1}{e^{\beta(\varepsilon-\mu)}\mp1}\qquad\Rightarrow\qquad\frac{f}{1\pm f}=e^{\beta(\mu-\varepsilon)}\quad , \tag{8.216}$$

[18] In this section f denotes the dimensionless occupation per spin state; densities are obtained by integrating it with $d^3k/(2\pi)^3$, with any spin degeneracy included separately.

which are the Boltzmann factors, which we have already shown to satisfy detailed balance. For the streaming term, we have

$$\begin{aligned} df^0 &= k_{\rm B}T\,\frac{\partial f^0}{\partial\varepsilon}\,d\left(\frac{\varepsilon-\mu}{k_{\rm B}T}\right) \\ &= k_{\rm B}T\,\frac{\partial f^0}{\partial\varepsilon}\left\{-\frac{d\mu}{k_{\rm B}T}-\frac{(\varepsilon-\mu)\,dT}{k_{\rm B}T^2}+\frac{d\varepsilon}{k_{\rm B}T}\right\} \\ &= -\frac{\partial f^0}{\partial\varepsilon}\left\{\frac{\partial\mu}{\partial\boldsymbol{r}}\cdot d\boldsymbol{r}+\frac{\varepsilon-\mu}{T}\,\frac{\partial T}{\partial\boldsymbol{r}}\cdot d\boldsymbol{r}-\frac{\partial\varepsilon}{\partial\boldsymbol{k}}\cdot d\boldsymbol{k}\right\} \quad , \end{aligned} \tag{8.217}$$

from which we read off

$$\begin{aligned} \frac{\partial f^0}{\partial\boldsymbol{r}} &= -\frac{\partial f^0}{\partial\varepsilon}\left\{\frac{\partial\mu}{\partial\boldsymbol{r}}+\frac{\varepsilon-\mu}{T}\,\frac{\partial T}{\partial\boldsymbol{r}}\right\} \\ \frac{\partial f^0}{\partial\boldsymbol{k}} &= \hbar\boldsymbol{v}\,\frac{\partial f^0}{\partial\varepsilon} \quad . \end{aligned} \tag{8.218}$$

Here we have assumed $\boldsymbol{V}=0$ for the unforced flow velocity of the electron fluid. The most important application is to the theory of electron transport in metals and semiconductors, in which case f^0 is the Fermi distribution. In this case, the quantum collision integral also receives a contribution from one-body scattering in the presence of an external potential $U(\boldsymbol{r})$, which is given by Fermi's Golden Rule:[19]

$$\begin{aligned} \left(\frac{\partial f(\boldsymbol{k})}{\partial t}\right)_{\rm coll} &= \frac{2\pi}{\hbar}\sum_{\boldsymbol{k}'\in\hat\Omega}|\langle\,\boldsymbol{k}'\,|\,U\,|\,\boldsymbol{k}\,\rangle|^2\,\big(f(\boldsymbol{k}')-f(\boldsymbol{k})\big)\,\delta\big(\varepsilon(\boldsymbol{k})-\varepsilon(\boldsymbol{k}')\big) \\ &= \frac{2\pi}{\hbar V}\int\limits_{\hat\Omega}\frac{d^3k'}{(2\pi)^3}\,|\,\hat U(\boldsymbol{k}-\boldsymbol{k}')|^2\,\big(f(\boldsymbol{k}')-f(\boldsymbol{k})\big)\,\delta\big(\varepsilon(\boldsymbol{k})-\varepsilon(\boldsymbol{k}')\big) \quad . \end{aligned} \tag{8.219}$$

The wavevectors are now restricted to the first Brillouin zone, and the dispersion $\varepsilon(\boldsymbol{k})$ is no longer the ballistic form $\varepsilon=\hbar^2\boldsymbol{k}^2/2m$ but rather the dispersion for electrons in a particular energy band of a solid[20]. Note that $f=f^0$ satisfies detailed balance with respect to one-body collisions as well[21].

In the presence of a weak electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$,[22] we have, within the relaxation time approximation, $f=f^0+\delta f$ with

$$\frac{\partial\,\delta f}{\partial t}-\frac{e}{\hbar c}\,\boldsymbol{v}\times\boldsymbol{B}\cdot\frac{\partial\,\delta f}{\partial\boldsymbol{k}}-\boldsymbol{v}\cdot\left[e\,\boldsymbol{\mathcal{E}}+\frac{\varepsilon-\mu}{T}\,\boldsymbol{\nabla}\,T\right]\frac{\partial f^0}{\partial\varepsilon}=-\frac{\delta f}{\tau} \quad , \tag{8.220}$$

[19] In general, for Bloch states, the matrix element $\langle\,n\boldsymbol{k}'\,|\,U\,|\,n\boldsymbol{k}\,\rangle$ is not simply a Fourier component $\hat U(\boldsymbol{k}-\boldsymbol{k}')/V$, but also involves the so-called cell functions. Here for simplicity we shall ignore the effect of the cell functions and approximate the Bloch states as plane waves.

[20] We neglect interband scattering here, which can be important in practical applications, but which is beyond the scope of these notes.

[21] The transition rate from $|\boldsymbol{k}'\rangle$ to $|\boldsymbol{k}\rangle$ is proportional to the matrix element and to the product $f'(1-f)$. The reverse process is proportional to $f(1-f')$. Subtracting these factors, one obtains $f'-f$, and therefore the nonlinear terms felicitously cancel in eqn. 8.219.

[22] The magnetic field $\boldsymbol{B}$ is not necessarily weak, but it is assumed to remain within the semiclassical regime, where Landau quantization can be neglected.

where $\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}(\phi - \mu/e) = \boldsymbol{E} + e^{-1}\boldsymbol{\nabla}\mu$ is minus the gradient of the 'electrochemical potential' $\phi - e^{-1}\mu$. In deriving the above equation, we have worked to lowest order in small quantities. This entails dropping terms like $\boldsymbol{v}\cdot\partial\,\delta f/\partial\boldsymbol{r}$ (higher order in spatial derivatives) and $\boldsymbol{E}\cdot\partial\,\delta f/\partial\boldsymbol{k}$ (both $\boldsymbol{E}$ and δf are assumed small). Typically τ is energy-dependent, *i.e.* $\tau = \tau\big(\varepsilon(\boldsymbol{k})\big)$.

We can use eqn. 8.220 to compute the electrical current $\boldsymbol{j}$ and the thermal current $\boldsymbol{j}_{\rm th}$,

$$\boldsymbol{j} = -2e\int\limits_{\hat\Omega}\frac{d^3k}{(2\pi)^3}\,\boldsymbol{v}\,\delta f \qquad , \qquad \boldsymbol{j}_{\rm th} = 2\int\limits_{\hat\Omega}\frac{d^3k}{(2\pi)^3}\,(\varepsilon-\mu)\,\boldsymbol{v}\,\delta f \quad . \tag{8.221}$$

Here the factor of 2 is from spin degeneracy of the electrons (we neglect Zeeman splitting).

In the presence of a time-independent temperature gradient and electric field, and taking $\boldsymbol{B} = 0$, the linearized Boltzmann equation in the relaxation time approximation has the solution

$$\delta f = -\tau(\varepsilon)\,\boldsymbol{v}\cdot\left(e\boldsymbol{\mathcal{E}} + \frac{\varepsilon-\mu}{T}\,\boldsymbol{\nabla}T\right)\left(-\frac{\partial f^0}{\partial\varepsilon}\right) \quad . \tag{8.222}$$

We now consider both the electrical current[23] $\boldsymbol{j}$ as well as the thermal current density $\boldsymbol{j}_{\rm th}$. One readily obtains

$$\begin{aligned}\boldsymbol{j} &= -2e\int\limits_{\hat\Omega}\frac{d^3k}{(2\pi)^3}\,\boldsymbol{v}\,\delta f \equiv L_{11}\,\boldsymbol{\mathcal{E}} - L_{12}\,\boldsymbol{\nabla}T\\ \boldsymbol{j}_{\rm th} &= 2\int\limits_{\hat\Omega}\frac{d^3k}{(2\pi)^3}\,(\varepsilon-\mu)\,\boldsymbol{v}\,\delta f \equiv L_{21}\,\boldsymbol{\mathcal{E}} - L_{22}\,\boldsymbol{\nabla}T\end{aligned} \tag{8.223}$$

where the *transport coefficients* L^{11} *etc.* are matrices:

$$\begin{aligned}L_{11}^{\alpha\beta} &= \frac{e^2}{4\pi^3\hbar}\int d\varepsilon\,\tau(\varepsilon)\left(-\frac{\partial f^0}{\partial\varepsilon}\right)\int dS_\varepsilon\,\frac{v^\alpha\,v^\beta}{|\boldsymbol{v}|}\\ L_{21}^{\alpha\beta} = TL_{12}^{\alpha\beta} &= -\frac{e}{4\pi^3\hbar}\int d\varepsilon\,\tau(\varepsilon)\,(\varepsilon-\mu)\left(-\frac{\partial f^0}{\partial\varepsilon}\right)\int dS_\varepsilon\,\frac{v^\alpha\,v^\beta}{|\boldsymbol{v}|}\\ L_{22}^{\alpha\beta} &= \frac{1}{4\pi^3\hbar T}\int d\varepsilon\,\tau(\varepsilon)\,(\varepsilon-\mu)^2\left(-\frac{\partial f^0}{\partial\varepsilon}\right)\int dS_\varepsilon\,\frac{v^\alpha\,v^\beta}{|\boldsymbol{v}|} \quad ,\end{aligned} \tag{8.224}$$

where dS_ε is a differential surface element in $\boldsymbol{k}$-space at fixed energy $\varepsilon(\boldsymbol{k}) = \varepsilon$, and

$$d^3k = \frac{d\varepsilon\,dS_\varepsilon}{|\partial\varepsilon/\partial\boldsymbol{k}|} = \frac{d\varepsilon\,dS_\varepsilon}{\hbar|\boldsymbol{v}|} \quad . \tag{8.225}$$

If we define the hierarchy of integral expressions

$$\mathcal{J}_n^{\alpha\beta} \equiv \frac{1}{4\pi^3\hbar}\int d\varepsilon\,\tau(\varepsilon)\,(\varepsilon-\mu)^n\left(-\frac{\partial f^0}{\partial\varepsilon}\right)\int dS_\varepsilon\,\frac{v^\alpha\,v^\beta}{|\boldsymbol{v}|} \tag{8.226}$$

[23] In this section we use $\boldsymbol{j}$ to denote electrical current, rather than particle number current as before.

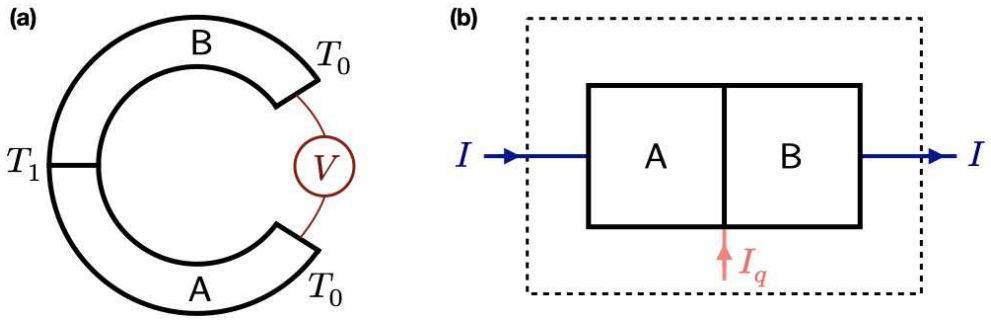


Figure 8.7: (a) A thermocouple is a junction formed of two dissimilar metals. With no electrical current passing, an electric field is generated in the presence of a temperature gradient, resulting in a voltage $V = V_{\rm A} - V_{\rm B} = (Q_{\rm A} - Q_{\rm B})(T_1 - T_0)$. (b) A sketch of a Peltier effect refrigerator. An electrical current I is passed through a junction between two dissimilar metals. If the dotted line represents the boundary of a thermally well-insulated body, then the body cools when $\Pi_{\rm B} > \Pi_{\rm A}$, in order to maintain a heat current balance at the junction, according to $I_{\rm th} = (\Pi_{\rm B} - \Pi_{\rm A})I$.

then we may write

$$L_{11}^{\alpha\beta} = e^2 \mathcal{J}_0^{\alpha\beta} \quad , \quad L_{21}^{\alpha\beta} = T L_{12}^{\alpha\beta} = -e\,\mathcal{J}_1^{\alpha\beta} \quad , \quad L_{22}^{\alpha\beta} = \frac{1}{T}\,\mathcal{J}_2^{\alpha\beta} \quad . \tag{8.227}$$

The linear relations in eqn. (8.223) may be recast in the following form:

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= \rho\,\boldsymbol{j} + Q\,\boldsymbol{\nabla} T \\ \boldsymbol{j}_{\rm th} &= \Pi\,\boldsymbol{j} - \kappa\,\boldsymbol{\nabla} T \quad , \end{aligned} \tag{8.228}$$

where the matrices ρ, Q, Π, and κ are given by

$$\rho = L_{11}^{-1} \quad , \quad Q = L_{11}^{-1} L_{12} \quad , \quad \Pi = L_{21} L_{11}^{-1} \quad , \quad \kappa = L_{22} - L_{21} L_{11}^{-1} L_{12} \quad , \tag{8.229}$$

or, in terms of the $\mathcal{J}_n$,

$$\rho = \frac{1}{e^2}\,\mathcal{J}_0^{-1} \quad , \quad Q = -\frac{1}{eT}\,\mathcal{J}_0^{-1}\,\mathcal{J}_1 \quad , \quad \Pi = -\frac{1}{e}\,\mathcal{J}_1\,\mathcal{J}_0^{-1} \quad , \quad \kappa = \frac{1}{T}\left(\mathcal{J}_2 - \mathcal{J}_1\,\mathcal{J}_0^{-1}\,\mathcal{J}_1\right) \quad , \tag{8.230}$$

These equations describe a wealth of transport phenomena:

- Electrical resistance ($\boldsymbol{\nabla} T = \boldsymbol{B} = 0$)
  An electrical current $\boldsymbol{j}$ will generate an electric field $\boldsymbol{\mathcal{E}} = \rho \boldsymbol{j}$, where ρ is the electrical resistivity.
- Peltier effect ($\boldsymbol{\nabla} T = \boldsymbol{B} = 0$)
  An electrical current $\boldsymbol{j}$ will generate a heat current $\boldsymbol{j}_{\rm th} = \Pi \boldsymbol{j}$, where Π is the Peltier coefficient.
- Thermal conduction ($\boldsymbol{j} = \boldsymbol{B} = 0$)
  A temperature gradient $\boldsymbol{\nabla} T$ gives rise to a heat current $\boldsymbol{j}_{\rm th} = -\kappa \boldsymbol{\nabla} T$, where κ is the thermal conductivity.

- Seebeck effect ($\boldsymbol{j} = \boldsymbol{B} = 0$)
  A temperature gradient $\boldsymbol{\nabla} T$ gives rise to an electric field $\boldsymbol{\mathcal{E}} = Q \boldsymbol{\nabla} T$, where Q is the Seebeck coefficient.

One practical way to measure the thermopower is to form a junction between two dissimilar metals, A and B. The junction is held at temperature T_1 and the other ends of the metals are held at temperature T_0. One then measures a voltage difference between the free ends of the metals – this is known as the Seebeck effect. Integrating the electric field from the free end of A to the free end of B gives[24]

$$V_{\rm A} - V_{\rm B} = \int_{\rm A}^{\rm B} \boldsymbol{\mathcal{E}} \cdot d\boldsymbol{l} = (Q_{\rm A} - Q_{\rm B})(T_1 - T_0) \quad . \tag{8.231}$$

What one measures here is really the difference in thermopowers of the two metals. For an absolute measurement of $Q_{\rm A}$, replace B by a superconductor ($Q = 0$ for a superconductor). A device which converts a temperature gradient into an emf is known as a *thermocouple*.

The Peltier effect has practical applications in refrigeration technology. Suppose an electrical current I is passed through a junction between two dissimilar metals, A and B. Due to the difference in Peltier coefficients, there will be a net heat current into the junction of $W = (\Pi_{\rm A} - \Pi_{\rm B})\, I$. Note that this is proportional to I, rather than the familiar I^2 result from Joule heating. The sign of W depends on the direction of the current. If a second junction is added, to make an ABA configuration, then heat absorbed at the first junction will be liberated at the second. [25]

8.9.2 The heat equation

We begin with the continuity equations for charge density ρ and energy density ε:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0 \qquad , \qquad \frac{\partial \varepsilon}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j}_\varepsilon = \boldsymbol{j} \cdot \boldsymbol{E} \quad , \tag{8.232}$$

where $\boldsymbol{E}$ is the electric field[26]. Now we invoke local thermodynamic equilibrium and write

$$\frac{\partial \varepsilon}{\partial t} = \left(\frac{\partial \varepsilon}{\partial n} \right)_T \frac{\partial n}{\partial t} + \left(\frac{\partial \varepsilon}{\partial T} \right)_n \frac{\partial T}{\partial t} = \left\{ \mu - T \left(\frac{\partial \mu}{\partial T} \right)_n \right\} \frac{\partial n}{\partial t} + c_V \frac{\partial T}{\partial t} \quad , \tag{8.233}$$

where n is the electron *number density*, $\rho = -en$ is the charge density, ε is the energy density, and c_V is the specific heat per unit volume at fixed density. We may now write

$$\begin{aligned} c_V \frac{\partial T}{\partial t} &= \frac{\partial \varepsilon}{\partial t} - \mu \frac{\partial n}{\partial t} + T \left(\frac{\partial \mu}{\partial T} \right)_n \frac{\partial n}{\partial t} \\ &= \boldsymbol{j} \cdot \boldsymbol{E} - \boldsymbol{\nabla} \cdot \boldsymbol{j}_{\rm th} + e^{-1} T \left(\frac{\partial \mu}{\partial T} \right)_n \boldsymbol{\nabla} \cdot \boldsymbol{j} \quad . \end{aligned} \tag{8.234}$$

[24] If the thermopowers are temperature-dependent, then $V_{\rm A} - V_{\rm B} = \int_{T_0}^{T_1} dT \, \big[Q_{\rm A}(T) - Q_{\rm B}(T) \big]$.

[25] To create a refrigerator, stick the cold junction inside a thermally insulated box and the hot junction outside the box.

[26] Note that it is $\boldsymbol{E} \cdot \boldsymbol{j}$ and *not* $\boldsymbol{\mathcal{E}} \cdot \boldsymbol{j}$ which is the source term in the energy continuity equation.

If the local density is time independent, charge continuity gives $\boldsymbol{\nabla}\cdot\boldsymbol{j}=0$ and we have

$$c_V\,\frac{\partial T}{\partial t}=\boldsymbol{j}\cdot\boldsymbol{\mathcal{E}}-\boldsymbol{\nabla}\cdot\boldsymbol{j}_{\rm th}\quad . \tag{8.235}$$

Finally, invoking $\boldsymbol{j}_{\rm th}=\sqcap\boldsymbol{j}-\kappa\boldsymbol{\nabla}T$, we see that if there is no electrical current ($\boldsymbol{j}=0$), we obtain the *heat equation*

$$c_V\,\frac{\partial T}{\partial t}=\frac{\partial}{\partial x^\alpha}\left(\kappa_{\alpha\beta}\,\frac{\partial T}{\partial x^\beta}\right)\quad . \tag{8.236}$$

Assuming isotropy, this results in a time scale $\tau_T=\mathcal{C}L^2c_V/\kappa$ for temperature diffusion, where L is a typical length scale and $\mathcal{C}$ is a numerical constant. For a cube of size L subjected to a sudden external temperature change, L is the side length and $\mathcal{C}=1/3\pi^2$ (solve by separation of variables).[27]

8.9.3 Calculation of Transport Coefficients

We will henceforth assume that sufficient crystalline symmetry exists (*e.g.* cubic symmetry) to render all the transport coefficients multiples of the identity matrix. Under such conditions, we may write $\mathcal{J}_n^{\alpha\beta}=\mathcal{J}_n\,\delta_{\alpha\beta}$ with

$$\mathcal{J}_n=\frac{1}{12\pi^3\hbar}\int d\varepsilon\,\tau(\varepsilon)\,(\varepsilon-\mu)^n\left(-\frac{\partial f^0}{\partial\varepsilon}\right)\int dS_\varepsilon\,|\boldsymbol{v}|\quad . \tag{8.237}$$

The low-temperature behavior is extracted using the Sommerfeld expansion,

$$\mathcal{I}\equiv\int\limits_{-\infty}^{\infty}d\varepsilon\,H(\varepsilon)\left(-\frac{\partial f^0}{\partial\varepsilon}\right)=\pi\mathcal{D}\csc(\pi\mathcal{D})\,H(\varepsilon)\Big|_{\varepsilon=\mu}=H(\mu)+\frac{\pi^2}{6}(k_{\rm B}T)^2\,H''(\mu)+\ldots\quad , \tag{8.238}$$

where $\mathcal{D}\equiv k_{\rm B}T\,\frac{\partial}{\partial\varepsilon}$ is a dimensionless differential operator[28].

Let us now perform some explicit calculations in the case of a parabolic band with an energy-independent scattering time τ. In this case, one readily finds

$$\mathcal{J}_n=\frac{\sigma_0}{e^2}\,\varepsilon_{\rm F}^{-3/2}\,\pi\mathcal{D}\csc\pi\mathcal{D}\,\varepsilon^{3/2}\,(\varepsilon-\mu)^n\Big|_{\varepsilon=\mu}\quad , \tag{8.239}$$

where $\sigma_0=ne^2\tau/m^*$. Note that

$$n=\frac{1}{3\pi^2}\left(\frac{2m^*\varepsilon_{\rm F}}{\hbar^2}\right)^{3/2} \tag{8.240}$$

and that $\varepsilon_{\rm F}$ and μ are related by

$$\varepsilon_{\rm F}^{3/2}=\pi\mathcal{D}\csc\pi\mathcal{D}\,\varepsilon^{3/2}\Big|_{\varepsilon=\mu}\quad . \tag{8.241}$$

[27] We assume constant isotropic coefficients, an initially uniform temperature, and all six faces held at the same new temperature after the change. The quoted time is the decay time of the slowest thermal mode.

[28] Remember that physically the fixed quantities are temperature and total carrier number density (or charge density, in the case of electron and hole bands), and *not* temperature and chemical potential. An equation of state relating n, μ, and T is then inverted to obtain $\mu(n,T)$, so that all results ultimately may be expressed in terms of n and T.

Thus,

$$\mathcal{J}_0 = \frac{\sigma_0}{e^2} \quad , \quad \mathcal{J}_1 = \frac{\sigma_0}{e^2}\frac{\pi^2}{2}\frac{(k_{\rm B}T)^2}{\mu} + \dots \quad , \quad \mathcal{J}_2 = \frac{\sigma_0}{e^2}\frac{\pi^2}{3}(k_{\rm B}T)^2 + \dots \quad , \tag{8.242}$$

from which we obtain the low-T results $\rho = \sigma_0^{-1}$,

$$Q = -\frac{\pi^2}{2}\,\frac{k_{\rm B}^2 T}{e\,\varepsilon_{\rm F}} \quad , \quad \kappa = \frac{\pi^2}{3}\,\frac{n\tau}{m^*}\,k_{\rm B}^2 T \quad , \tag{8.243}$$

and of course $\sqcap = TQ$. The predicted universal ratio

$$\frac{\kappa}{\sigma T} = \frac{\pi^2}{3}\,(k_{\rm B}/e)^2 = 2.45 \times 10^{-8}\,{\rm V}^2\,{\rm K}^{-2} \quad , \tag{8.244}$$

is known as the *Wiedemann-Franz law*. Note also that our result for the thermopower is unambiguously negative. In actuality, several nearly free electron metals have positive low-temperature thermopowers (Cs and Li, for example). What went wrong? The constant τ, parabolic-band approximation is too restrictive. Realistic band structure and energy-dependence of scattering time can alter or reverse the thermopower, while phonon drag can supply an additional contribution.

8.9.4 Onsager Relations

Transport phenomena are described in general by a set of linear relations,

$$J_i = L_{ik}\,F_k \quad , \tag{8.245}$$

where the $\{F_k\}$ are *generalized forces* and the $\{J_i\}$ are *generalized currents*. Moreover, to each force F_i corresponds a unique conjugate current J_i, such that the rate of internal entropy production is

$$\dot{S} = \sum_i F_i\,J_i \quad . \tag{8.246}$$

The Onsager relations (also known as *Onsager reciprocity*) state that

$$L_{ik}(\boldsymbol{B}) = \eta_i\,\eta_k\,L_{ki}(-\boldsymbol{B}) \quad , \tag{8.247}$$

where η_i describes the parity of J_i under time reversal:

$$J_i^T = \eta_i\,J_i \quad , \tag{8.248}$$

where J_i^T is the time reverse of J_i. To justify the Onsager relations requires a microscopic description of our nonequilibrium system.

The Onsager relations have some remarkable consequences. For example, they require, for $\boldsymbol{B} = 0$, that the thermal conductivity tensor κ_{ij} of any crystal must be symmetric, independent of the crystal structure. In general, this result does not follow from considerations of crystalline symmetry. It also requires that for every 'off-diagonal' transport phenomenon, *e.g.* the Seebeck effect, there exists a distinct corresponding phenomenon, *e.g.* the Peltier effect.

For the transport coefficients studied, Onsager reciprocity means that in the presence of an external magnetic field,

$$\rho_{\alpha\beta}(\boldsymbol{B}) = \rho_{\beta\alpha}(-\boldsymbol{B}) \quad , \quad \kappa_{\alpha\beta}(\boldsymbol{B}) = \kappa_{\beta\alpha}(-\boldsymbol{B}) \quad , \quad \sqcap_{\alpha\beta}(\boldsymbol{B}) = T\,Q_{\beta\alpha}(-\boldsymbol{B}) \quad . \tag{8.249}$$

Let's consider an isotropic system in a weak magnetic field, and expand the transport coefficients to first order in $\boldsymbol{B}$:

$$\begin{aligned}
\rho_{\alpha\beta}(\boldsymbol{B}) &= \rho\,\delta_{\alpha\beta} + \nu\,\epsilon_{\alpha\beta\gamma}\,B^\gamma \\
\kappa_{\alpha\beta}(\boldsymbol{B}) &= \kappa\,\delta_{\alpha\beta} + \varpi\,\epsilon_{\alpha\beta\gamma}\,B^\gamma \\
Q_{\alpha\beta}(\boldsymbol{B}) &= Q\,\delta_{\alpha\beta} + \zeta\,\epsilon_{\alpha\beta\gamma}\,B^\gamma \\
\sqcap_{\alpha\beta}(\boldsymbol{B}) &= \sqcap\,\delta_{\alpha\beta} + \theta\,\epsilon_{\alpha\beta\gamma}B^\gamma \quad .
\end{aligned} \tag{8.250}$$

Onsager reciprocity requires $\sqcap = T\,Q$ and $\theta = T\,\zeta$. We can now write

$$\begin{aligned}
\boldsymbol{\mathcal{E}} &= \rho\,\boldsymbol{j} + \nu\,\boldsymbol{j}\times\boldsymbol{B} + Q\,\boldsymbol{\nabla}T + \zeta\,\boldsymbol{\nabla}T\times\boldsymbol{B} \\
\boldsymbol{j}_{\rm th} &= \sqcap\,\boldsymbol{j} + \theta\,\boldsymbol{j}\times\boldsymbol{B} - \kappa\,\boldsymbol{\nabla}T - \varpi\,\boldsymbol{\nabla}T\times\boldsymbol{B} \quad .
\end{aligned} \tag{8.251}$$

There are several new phenomena lurking:

- Hall effect ($\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = j_y = 0$)
  An electrical current $\boldsymbol{j} = j_x\,\boldsymbol{x}$ and a field $\boldsymbol{B} = B_z\,\hat{\boldsymbol{z}}$ yield an electric field $\boldsymbol{\mathcal{E}}$. The Hall coefficient is $R_{\rm H} = \mathcal{E}_y/j_x\,B_z = -\nu$.
- Ettingshausen effect ($\frac{\partial T}{\partial x} = j_y = j_{{\rm th},y} = 0$)
  An electrical current $\boldsymbol{j} = j_x\,\boldsymbol{x}$ and a field $\boldsymbol{B} = B_z\,\hat{\boldsymbol{z}}$ yield a temperature gradient $\frac{\partial T}{\partial y}$. The Ettingshausen coefficient is $P = \frac{\partial T}{\partial y}/j_x\,B_z = -\theta/\kappa$.
- Nernst effect ($j_x = j_y = \frac{\partial T}{\partial y} = 0$)
  A temperature gradient $\boldsymbol{\nabla}T = \frac{\partial T}{\partial x}\,\boldsymbol{x}$ and a field $\boldsymbol{B} = B_z\,\hat{\boldsymbol{z}}$ yield an electric field $\boldsymbol{\mathcal{E}}$. The Nernst coefficient is $\Lambda = \mathcal{E}_y/\frac{\partial T}{\partial x}\,B_z = -\zeta$.
- Righi-Leduc effect ($j_x = j_y = j_{{\rm th},y} = 0$)
  A temperature gradient $\boldsymbol{\nabla}T = \frac{\partial T}{\partial x}\,\boldsymbol{x}$ and a field $\boldsymbol{B} = B_z\,\hat{\boldsymbol{z}}$ yield an orthogonal temperature gradient $\frac{\partial T}{\partial y}$. The Righi-Leduc coefficient is $\mathcal{L} = \frac{\partial T}{\partial y}/\frac{\partial T}{\partial x}B_z = \varpi/\kappa$.

8.10 Appendix I: The BBGKY Hierarchy and the Boltzmann Equation

The procedure by which the Boltzmann equation for a gas is obtained from N-particle Hamiltonian dynamics of its constituent particles is known as the *Boltzmann-Grad limit*. An excellent source, which we follow here, is Kardar (Particles), §3.3. The formal derivation of the Boltzmann equation from reversible hard sphere dynamics in the low-density limit is generally credited to R. Lanford, "Time Evolution of Large Classical Systems" in *Dynamical Systems, Theory and Applications* (Springer Lecture Notes in Physics, 1975).

8.10.1 BBGKY hierarchy

We start with the time-dependent distribution function, $\varrho_N(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N, t)$, on N-particle phase space, which is assumed normalized according to

$$\int \prod_{j=1}^{N} d\mu_j \; \varrho_N(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N, t) = 1 \quad , \tag{8.252}$$

where in this section we write $d\mu_j \equiv d^d x_j \, d^d p_j$ without the h^{-d} factor introduced in §8.2. We also assume $d = 3$. We define the s-particle distribution function,

$$f_s\big(\{\boldsymbol{x}_i\}, \{\boldsymbol{p}_i\}, t\big) = \frac{N!}{(N-s)!} \int \prod_{i=s+1}^{N} d\mu_i \; \varrho_N\big(\{\boldsymbol{x}_j\}, \{\boldsymbol{p}_j\}, t\big) \equiv \frac{N!}{(N-s)!} \, \varrho_s\big(\{\boldsymbol{x}_i\}, \{\boldsymbol{p}_i\}, t\big) \quad . \tag{8.253}$$

We adopt the notation that $\{\boldsymbol{x}_j\} \equiv \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_s\}$ when it appears as an argument of a function f_s or ϱ_s, and similarly for $\{\boldsymbol{p}_j\}$. The dynamics of ϱ_N are given by Liouville's equation, $\partial_t \, \varrho_N + \{\varrho_N, H_N\} = 0$, where

$$H_N\big(\{\boldsymbol{x}_j\}, \{\boldsymbol{p}_j\}\big) = \sum_{j=1}^{N} \left[\frac{\boldsymbol{p}_j^2}{2m} + v(\boldsymbol{x}_j) \right] + \sum_{i<j}^{N} u(\boldsymbol{x}_i - \boldsymbol{x}_j) \quad , \tag{8.254}$$

where $v(\boldsymbol{x})$ is an external potential. We may write, for any $k \in \{1, \ldots, N\}$,

$$H_N = H_k + \bar{H}_k + H'_k \quad , \tag{8.255}$$

where

$$\begin{aligned} H_k &= \sum_{j=1}^{k} \left[\frac{\boldsymbol{p}_j^2}{2m} + v(\boldsymbol{x}_j) \right] + \sum_{0<i<j}^{k} u(\boldsymbol{x}_i - \boldsymbol{x}_j) \\ \bar{H}_k &= \sum_{j=k+1}^{N} \left[\frac{\boldsymbol{p}_j^2}{2m} + v(\boldsymbol{x}_j) \right] + \sum_{k<i<j}^{N} u(\boldsymbol{x}_i - \boldsymbol{x}_j) \\ H'_k &= \sum_{i=1}^{k} \sum_{j=k+1}^{N} u(\boldsymbol{x}_i - \boldsymbol{x}_j) \quad . \end{aligned} \tag{8.256}$$

We further assume that $u(\boldsymbol{x}_i - \boldsymbol{x}_j) = u(|\boldsymbol{x}_i - \boldsymbol{x}_j|)$ is a central potential. The dynamics of ϱ_s are then

$$\frac{\partial \varrho_s}{\partial t} = \int \prod_{j=s+1}^{N} d\mu_j \; \frac{\partial \varrho_N}{\partial t} = -\int \prod_{j=s+1}^{N} d\mu_j \; \big\{ \varrho_N, H_s + \bar{H}_s + H'_s \big\} \quad . \tag{8.257}$$

We now evaluate each of the three contributions to the Poisson bracket.

The first contribution is easy:

$$\left[\frac{\partial \varrho_s}{\partial t} \right]_{(1)} = -\int \prod_{j=s+1}^{N} d\mu_j \; \big\{ \varrho_N, H_s \big\} = -\big\{ \varrho_s, H_s \big\} \quad . \tag{8.258}$$

The second contribution is

$$\begin{aligned}\left[\frac{\partial \varrho_s}{\partial t}\right]_{(2)} &= -\int\prod_{j=s+1}^{N} d\mu_j \{\varrho_N, \bar{H}_s\} \\ &= \int\prod_{j=s+1}^{N} d\mu_j \sum_{k=s+1}^{N}\left[\frac{\partial \varrho_N}{\partial \boldsymbol{p}_k}\cdot\frac{\partial \bar{H}_s}{\partial \boldsymbol{x}_k} - \frac{\partial \varrho_N}{\partial \boldsymbol{x}_k}\cdot\frac{\partial \bar{H}_s}{\partial \boldsymbol{p}_k}\right] \\ &= \int\prod_{j=s+1}^{N} d\mu_j \sum_{k=s+1}^{N}\left[\frac{\partial \varrho_N}{\partial \boldsymbol{p}_k}\cdot\overbrace{\left(\frac{\partial v(\boldsymbol{x}_k)}{\partial \boldsymbol{x}_k} + \sum_{\substack{\ell=s+1\\ \ell\neq k}}^{N}\frac{\partial u(\boldsymbol{x}_k-\boldsymbol{x}_\ell)}{\partial \boldsymbol{x}_k}\right)}^{\text{independent of } \boldsymbol{p}_k} - \frac{\partial \varrho_N}{\partial \boldsymbol{x}_k}\cdot\frac{\boldsymbol{p}_k}{m}\right] = 0 \quad .\end{aligned} \tag{8.259}$$

Here we integrate by parts on both terms inside the square brackets. The term in rounded brackets is independent of $\boldsymbol{p}_k$, while $\boldsymbol{p}_k/m$ is independent of $\boldsymbol{x}_k$. Thus, the second contribution vanishes.

The third contribution is

$$\begin{aligned}\left[\frac{\partial \varrho_s}{\partial t}\right]_{(3)} &= -\int\prod_{j=s+1}^{N} d\mu_j \left\{\varrho_N, H'_s\right\} \\ &= \int\prod_{j=s+1}^{N} d\mu_j \sum_{k=1}^{s}\sum_{\ell=s+1}^{N}\left(\frac{\partial \rho_N}{\partial \boldsymbol{p}_k}\cdot\frac{\partial u(\boldsymbol{x}_k-\boldsymbol{x}_\ell)}{\partial \boldsymbol{x}_k} + \frac{\partial \rho_N}{\partial \boldsymbol{p}_\ell}\cdot\frac{\partial u(\boldsymbol{x}_k-\boldsymbol{x}_\ell)}{\partial \boldsymbol{x}_\ell}\right) \\ &= (N-s)\int d\mu_{s+1}\sum_{k=1}^{s}\frac{\partial u(\boldsymbol{x}_k-\boldsymbol{x}_{s+1})}{\partial \boldsymbol{x}_k}\cdot\frac{\partial}{\partial \boldsymbol{p}_k}\left(\int\prod_{\ell=s+2}^{N} d\mu_\ell\, \varrho_N\right)\end{aligned} \tag{8.260}$$

Thus, we arrive at

$$\frac{\partial \varrho_s}{\partial t} + \left\{\rho_s, H_s\right\} = (N-s)\int d\mu_{s+1}\sum_{k=1}^{s}\frac{\partial u(\boldsymbol{x}_k-\boldsymbol{x}_{s+1})}{\partial \boldsymbol{x}_k}\cdot\frac{\partial \varrho_{s+1}}{\partial \boldsymbol{p}_k} \quad , \tag{8.261}$$

or

$$\frac{\partial f_s}{\partial t} + \left\{f_s, H_s\right\} = \int d\mu_{s+1}\sum_{k=1}^{s}\frac{\partial u(\boldsymbol{x}_k-\boldsymbol{x}_{s+1})}{\partial \boldsymbol{x}_k}\cdot\frac{\partial f_{s+1}}{\partial \boldsymbol{p}_k} \quad . \tag{8.262}$$

This is the BBGKY hierarchy.

8.10.2 Boltzmann equation

At each level of the hierarchy, we may identify three types of terms, each of which is associated with its own time scale. The size of each type is roughly inversely proportional to its characteristic time scale[29]. The first type of term arises from the noninteracting terms in H_s, *i.e.* the one-body terms, which do not involve the interaction potential $u(\boldsymbol{x}_i - \boldsymbol{x}_j)$. The time scale associated with these terms behaves as

[29] See Kardar §3.3 for a more detailed discussion.

$\tau_v \sim L/V$, where V is a characteristic velocity and L the linear system size. This extrinsic scale becomes arbitrarily large as $L \to \infty$. The second type of term are those on the LHS of the equation for ϱ_s. Assuming $u(\boldsymbol{x})$ is short-ranged, such as a van der Waals or Lennard-Jones potential, the typical distance scale is $d \sim 1$ Å, which entails a collision time $\tau_{\rm c} \sim 10^{-12}$ s. Finally, the third type at each level are terms on the RHS which are proportional to the interaction potential u. Because the distribution function f_{s+1} enters, these terms require an additional particle be present, and the time constant is longer than $\tau_{\rm c}$ by a factor $1/nd^3 \sim 10^4$. During a short binary encounter, the probability of an additional particle participating is small in the dilute limit. We are then licensed to neglect the f_3 term when resolving the two-particle collision, while retaining the accumulated effect of successive binary collisions in the one-particle kinetic equation.

At level $s = 2$, the equation for f_2 is

$$\left\{ \frac{\partial}{\partial t} + \frac{\boldsymbol{p}_1}{m} \cdot \frac{\partial}{\partial \boldsymbol{x}_1} + \frac{\boldsymbol{p}_2}{m} \cdot \frac{\partial}{\partial \boldsymbol{x}_2} - \frac{\partial u(\boldsymbol{x}_2 - \boldsymbol{x}_1)}{\partial \boldsymbol{x}_1} \cdot \frac{\partial}{\partial \boldsymbol{p}_1} - \frac{\partial u(\boldsymbol{x}_2 - \boldsymbol{x}_1)}{\partial \boldsymbol{x}_2} \cdot \frac{\partial}{\partial \boldsymbol{p}_2} \right\} f_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t) = 0 \quad . \tag{8.263}$$

Because the incoming distribution varies slowly compared with the collision time $\tau_{\rm c}$, we approximate the two-particle encounter by a stationary scattering problem with all macroscopic parameters held fixed. In the presence of slowly time-varying parameters such as the temperature $T(\boldsymbol{r}, t)$ or chemical potential $\mu(\boldsymbol{r}, t)$, or in the presence of a time-dependent external force $\boldsymbol{F}(\boldsymbol{r}, t)$, the distribution f_2 will evolve, but on a longer time scale associated with these respective variations. Therefore we can fix the time t and freeze these slow changes and ask about the steady state distribution $f_2^{\rm ss}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t)$. In discussing the 'instantaneous steady state' – a seemingly nonsensical collocation which however has meaning given the aforementioned separation of time scales – we suppress for the moment the t label and we define $\phi(\boldsymbol{X}, \boldsymbol{x}, \boldsymbol{p}_1, \boldsymbol{p}_2) \equiv f_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{p}_1, \boldsymbol{p}_2)$, where $\boldsymbol{X} \equiv \frac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_2)$ is the center-of-mass coordinate and $\boldsymbol{x} \equiv \boldsymbol{x}_2 - \boldsymbol{x}_1$ is the relative coordinate. We then have

$$\left\{ \left(\frac{\boldsymbol{p}_1 + \boldsymbol{p}_2}{2m} \right) \cdot \frac{\partial}{\partial \boldsymbol{X}} + \left(\frac{\boldsymbol{p}_2 - \boldsymbol{p}_1}{m} \right) \cdot \frac{\partial}{\partial \boldsymbol{x}} + \boldsymbol{\nabla} u(\boldsymbol{x}) \cdot \left(\frac{\partial}{\partial \boldsymbol{p}_1} - \frac{\partial}{\partial \boldsymbol{p}_2} \right) \right\} \phi(\boldsymbol{X}, \boldsymbol{x}, \boldsymbol{p}_1, \boldsymbol{p}_2) = 0 \quad . \tag{8.264}$$

We recognize this as a linear partial differential equation in the 12 variables $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{p}_1, \boldsymbol{p}_2\}$, which, in principle, may be solved by the method of characteristics. This entails introducing a scalar variable τ and solving the coupled ODEs

$$\frac{d\boldsymbol{X}}{d\tau} = \frac{\boldsymbol{p}_1 + \boldsymbol{p}_2}{2m} \quad , \quad \frac{d\boldsymbol{x}}{d\tau} = \frac{\boldsymbol{p}_2 - \boldsymbol{p}_1}{m} \quad , \quad \frac{d\boldsymbol{p}_1}{d\tau} = \boldsymbol{\nabla} u(\boldsymbol{x}) \quad , \quad \frac{d\boldsymbol{p}_2}{d\tau} = -\boldsymbol{\nabla} u(\boldsymbol{x}) \quad . \tag{8.265}$$

This defines a path in the 12-dimensional space. Along such a path,

$$\frac{d\phi\big(\boldsymbol{X}(\tau), \boldsymbol{x}(\tau), \boldsymbol{p}_1(\tau), \boldsymbol{p}_2(\tau)\big)}{d\tau} = \frac{\partial \phi}{\partial \boldsymbol{X}} \cdot \frac{d\boldsymbol{X}}{d\tau} + \frac{\partial \phi}{\partial \boldsymbol{x}} \cdot \frac{d\boldsymbol{x}}{d\tau} + \frac{\partial \phi}{\partial \boldsymbol{p}_1} \cdot \frac{d\boldsymbol{p}_1}{d\tau} + \frac{\partial \phi}{\partial \boldsymbol{p}_2} \cdot \frac{d\boldsymbol{p}_2}{d\tau} = 0 \quad . \tag{8.266}$$

The $N = 12$ variable PDE of eqn. 8.264 thus has been converted into a system of $N + 1 = 13$ coupled ODEs of eqns. 8.265 and 8.266. We see that τ functions as a time variable, and the coupled ODEs of eqn. 8.265 are simply Hamilton's equations of motion. Note that $\boldsymbol{P} \equiv \boldsymbol{p}_1 + \boldsymbol{p}_2$ satisfies $d\boldsymbol{P}/dt = 0$ and we have the familiar CM motion $\boldsymbol{X}(\tau) = \boldsymbol{X}(0) + \boldsymbol{P}\tau/M$, with $M = 2m$ the total mass. Initial conditions

for the N variables $\boldsymbol{w} = \{w_1, \ldots, w_N\}$ at $\tau = 0$ are specified by fixing a hypersurface parameterized by $N-1$ variables $\boldsymbol{\zeta} = \{\zeta_1, \ldots, \zeta_{N-1}\}$, where[30]

$$w_j(\tau = 0) = h_j(\zeta_1, \ldots, \zeta_{N-1}) \quad , \quad \phi(\tau = 0) = F(\zeta_1, \ldots, \zeta_{N-1}) \quad , \tag{8.267}$$

and where the $\{h_j(\boldsymbol{\zeta})\}$ define the hypersurface. A simple way to think of this is to define the hypersurface by the relation $w_N = 0$, and take $w_j(\tau = 0) = \zeta_j$ for $j \in \{1, \ldots, N-1\}$, *i.e.* $h_j(\boldsymbol{\zeta}) = \zeta_j$. Each characteristic is labeled by an $(N-1)$-tuple $\boldsymbol{\zeta}$, and the solution along a characteristic is $\phi(\boldsymbol{w}) = \phi(\tau; \boldsymbol{\zeta})$.

Now we turn to level $s = 1$ of the hierarchy, where we have

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}_1}{m} \cdot \frac{\partial f}{\partial \boldsymbol{x}_1} - \boldsymbol{\nabla} v(\boldsymbol{x}_1) \cdot \frac{\partial f}{\partial \boldsymbol{p}_1} = \int d\mu_2 \, \frac{\partial u(\boldsymbol{x}_2 - \boldsymbol{x}_1)}{\partial \boldsymbol{x}_1} \cdot \left(\frac{\partial f_2}{\partial \boldsymbol{p}_1} - \frac{\partial f_2}{\partial \boldsymbol{p}_2} \right) \tag{8.268}$$

We now invoke the relation

$$-\boldsymbol{\nabla} u(\boldsymbol{x}) \cdot \left(\frac{\partial f_2}{\partial \boldsymbol{p}_1} - \frac{\partial f_2}{\partial \boldsymbol{p}_2} \right) = \left(\frac{\boldsymbol{p}_1 + \boldsymbol{p}_2}{2m} \right) \cdot \frac{\partial f_2}{\partial \boldsymbol{X}} + \left(\frac{\boldsymbol{p}_2 - \boldsymbol{p}_1}{m} \right) \cdot \frac{\partial f_2}{\partial \boldsymbol{x}} \quad , \tag{8.269}$$

leading us to

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}_1}{m} \cdot \frac{\partial f}{\partial \boldsymbol{x}_1} - \boldsymbol{\nabla} v(\boldsymbol{x}_1) \cdot \frac{\partial f}{\partial \boldsymbol{p}_1} = \int d^3 p_2 \int d^3 x \left(\frac{\boldsymbol{p}_2 - \boldsymbol{p}_1}{m} \right) \cdot \frac{\partial f_2(\boldsymbol{x}_1, \boldsymbol{x}_1 + \boldsymbol{x}, \boldsymbol{p}_1, \boldsymbol{p}_2, t)}{\partial \boldsymbol{x}} \quad , \tag{8.270}$$

where we have restored the time label t. In arriving at this last equation, note that we have dropped terms proportional to $\partial f_2 / \partial \boldsymbol{X}$. This is justified by the presumption that the center-of-mass dynamics are relatively slow. Note also that on the RHS we are expressing the spatial coordinates in terms of $\boldsymbol{x}_1$ and $\boldsymbol{x}$, hence we have substituted $\boldsymbol{x}_2 = \boldsymbol{x}_1 + \boldsymbol{x}$ in the second vector argument of f_2.

Consider the integral on the RHS of eqn. 8.270 over the relative coordinate $\boldsymbol{x}$. Following Kardar, at fixed $\boldsymbol{x}_1$, $\boldsymbol{p}_1$, and $\boldsymbol{p}_2$, we choose a coordinate system for $\boldsymbol{x}$ where one axis is parallel to $\hat{\boldsymbol{n}} = (\boldsymbol{p}_2 - \boldsymbol{p}_1)/|\boldsymbol{p}_2 - \boldsymbol{p}_1|$ and write

$$\boldsymbol{x} = x_\parallel \, \hat{\boldsymbol{n}} + \boldsymbol{b} \quad , \tag{8.271}$$

where $\hat{\boldsymbol{n}} \cdot \boldsymbol{b} = 0$. The vector $\boldsymbol{b}$ is known as the *impact parameter vector*. We then have

$$\left(\frac{df}{dt} \right)_{\rm coll} = \int d^3 p_2 \int d^2 b \left| \frac{\boldsymbol{p}_1 - \boldsymbol{p}_2}{m} \right| \left\{ \tilde{f}_2^+(\boldsymbol{x}_1, \boldsymbol{b}, \boldsymbol{p}_1, \boldsymbol{p}_2, t) - \tilde{f}_2^-(\boldsymbol{x}_1, \boldsymbol{b}, \boldsymbol{p}_1, \boldsymbol{p}_2, t) \right\} \quad , \tag{8.272}$$

where

$$\tilde{f}_2^\pm(\boldsymbol{x}_1, \boldsymbol{b}, \boldsymbol{p}_1, \boldsymbol{p}_2, t) \equiv f_2(\boldsymbol{x}_1, \boldsymbol{x}_1 + x_\parallel^\pm \, \hat{\boldsymbol{n}} + \boldsymbol{b}, \boldsymbol{p}_1, \boldsymbol{p}_2, t) \quad , \tag{8.273}$$

and where $x_\parallel^\pm$ is the value of $\hat{\boldsymbol{n}} \cdot \boldsymbol{x} = \hat{\boldsymbol{n}} \cdot (\boldsymbol{x}_2 - \boldsymbol{x}_1)$ after $(+)$ or before $(-)$ the collision. We assume the potential $u(\boldsymbol{x})$ is sufficiently short-ranged that $x_\parallel^\pm$ may be evaluated a short distance from the collision center $\boldsymbol{x} = 0$. In a scattering process, the momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ prior to the scattering event evolve into $\boldsymbol{p}_1'$ and $\boldsymbol{p}_2'$ afterward. The relation between the incoming and outgoing momenta is given by eqn. 8.63, *i.e.*

$$\begin{aligned} \boldsymbol{p}_1' &= \tfrac{1}{2} \big(\boldsymbol{p}_1 + \boldsymbol{p}_2 + |\boldsymbol{p}_1 - \boldsymbol{p}_2| \, \hat{\boldsymbol{\Omega}} \big) \\ \boldsymbol{p}_2' &= \tfrac{1}{2} \big(\boldsymbol{p}_1 + \boldsymbol{p}_2 - |\boldsymbol{p}_1 - \boldsymbol{p}_2| \, \hat{\boldsymbol{\Omega}} \big) \quad , \end{aligned} \tag{8.274}$$

[30] The initial hypersurface must be *noncharacteristic*, so the characteristics cross it transversely.

where $\hat{\boldsymbol{\Omega}} = \hat{\boldsymbol{\Omega}}(\boldsymbol{b})$ is a unit vector which is determined by the impact parameter vector $\boldsymbol{b}$. Assuming the system is time-reversal invariant, we may relate $\tilde{f}_2^+$ and $\tilde{f}_2^-$ by

$$\tilde{f}_2^+(\boldsymbol{x}_1, \boldsymbol{b}, \boldsymbol{p}_1, \boldsymbol{p}_2, t) = \tilde{f}_2^-(\boldsymbol{x}_1, \boldsymbol{b}, \boldsymbol{p}_1', \boldsymbol{p}_2', t) \quad . \tag{8.275}$$

This suggests that we could write the collision integral in either of two ways:

$$\begin{aligned}\left(\frac{df}{dt}\right)_{\text{coll},-} &= \int d^3p_2 \int d\Omega \, \frac{d\sigma}{d\Omega} \, |\boldsymbol{v}_1 - \boldsymbol{v}_2| \left\{ \tilde{f}_2^-(\boldsymbol{x}_1, \boldsymbol{b}, \boldsymbol{p}_1', \boldsymbol{p}_2', t) - \tilde{f}_2^-(\boldsymbol{x}_1, \boldsymbol{b}, \boldsymbol{p}_1, \boldsymbol{p}_2, t) \right\} \\ \left(\frac{df}{dt}\right)_{\text{coll},+} &= \int d^3p_2 \int d\Omega \, \frac{d\sigma}{d\Omega} \, |\boldsymbol{v}_1 - \boldsymbol{v}_2| \left\{ \tilde{f}_2^+(\boldsymbol{x}_1, \boldsymbol{b}, \boldsymbol{p}_1, \boldsymbol{p}_2, t) - \tilde{f}_2^+(\boldsymbol{x}_1, \boldsymbol{b}, \boldsymbol{p}_1', \boldsymbol{p}_2', t) \right\} \quad .\end{aligned} \tag{8.276}$$

Here we have also written

$$d^2b = \frac{d\sigma}{d\Omega} \, d\Omega \quad , \tag{8.277}$$

where $d\Omega$ is the differential solid angle of the scattering process. The expression $d\sigma/d\Omega$, which should be familiar from both classical mechanics and quantum mechanics, is the *differential scattering cross-section*. We assume there is a unique relationship between the scattering direction $\hat{\boldsymbol{\Omega}}(\boldsymbol{b})$ and the impact parameter vector $\boldsymbol{b}$, which is to say that the $\boldsymbol{b}$ plane can be mapped to the unit sphere $\hat{\boldsymbol{\Omega}}(\boldsymbol{b})$ in a one-to-one fashion. For a short-range potential, appreciable deflections arise only from impact parameters comparable to the interaction range $r_{\rm c}$. Trajectories with much larger impact parameter $|\boldsymbol{b}| \gg r_{\rm c}$ are essentially undeflected. At long last, we make the assumption,

$$\tilde{f}_2^{\pm}(\boldsymbol{x}_1, \boldsymbol{b}, \boldsymbol{p}_1, \boldsymbol{p}_2, t) \approx f_1(\boldsymbol{x}_1, \boldsymbol{p}_1, t) \times f_1(\boldsymbol{x}_2, \boldsymbol{p}_2, t) \quad . \tag{8.278}$$

This is known as the *Stosszahlansatz* or the *hypothesis of molecular chaos*, and amounts to assuming that the velocities of colliding particles are uncorrelated. But are they uncorrelated before ($-$) or after ($+$) the collision? The answer makes a huge difference! It determines which of the candidate collision integrals in eqn. 8.276 is correct.

Our Boltzmann equation for the one-body distribution $f(\boldsymbol{r}, \boldsymbol{p}, t)$ is now given by

$$\begin{aligned}\left(\frac{df}{dt}\right)_{\text{coll}} &= \pm \int d^3p_1 \int d\Omega \, \frac{d\sigma}{d\Omega} \left| \frac{\boldsymbol{p} - \boldsymbol{p}_1}{m} \right| \left\{ f(\boldsymbol{r}, \boldsymbol{p}, t) f(\boldsymbol{r}, \boldsymbol{p}_1, t) - f(\boldsymbol{r}, \boldsymbol{p}', t) f(\boldsymbol{r}, \boldsymbol{p}_1', t) \right\} \\ &= \pm \int d^3p_1 \int d^3p' \int d^3p_1' \, w(\boldsymbol{p}, \boldsymbol{p}_1 \,|\, \boldsymbol{p}', \boldsymbol{p}_1') \left\{ f(\boldsymbol{r}, \boldsymbol{p}, t) f(\boldsymbol{r}, \boldsymbol{p}_1, t) - f(\boldsymbol{r}, \boldsymbol{p}', t) f(\boldsymbol{r}, \boldsymbol{p}_1', t) \right\}\end{aligned} \tag{8.279}$$

where we have used eqn. 8.36 in the last line. According to the results of §8.3.7, we see that choosing $\tilde{f}_2^+$ in eqn. 8.278 results in a generically *positive* source for Boltzmann's $\mathcal{H}$ density, h, which is to say $\partial_t h + \boldsymbol{\nabla} \cdot \boldsymbol{j} \geq 0$, so for this choice we have an "anti $\mathcal{H}$-theorem". Thermodynamically, this means that entropy of an isolated system *decreases* (since $S = -k_{\rm B}\mathcal{H}$), which is the exact opposite of what the second law of thermodynamics requires. Thus, the correct choice is to take $\tilde{f}_2^-$ in eqn. 8.278, in which case our Boltzmann equation takes the final form[31]

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \frac{\partial v}{\partial \boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = \int d^3p_1 \int d^3p' \int d^3p_1' \, w(\boldsymbol{p}, \boldsymbol{p}_1 \,|\, \boldsymbol{p}', \boldsymbol{p}_1') \left\{ f(\boldsymbol{r}, \boldsymbol{p}', t) f(\boldsymbol{r}, \boldsymbol{p}_1', t) - f(\boldsymbol{r}, \boldsymbol{p}, t) f(\boldsymbol{r}, \boldsymbol{p}_1, t) \right\} \quad . \tag{8.280}$$

[31] Note also that if we were to impose the *Stosszahlansatz* on each term inside the curly brackets on the RHS of eqn. 8.272, the collision integral would vanish!

Why is it necessary to invoke the *Stosszahlansatz* for particles which are just about to collide, but *not* for particles which have just collided? This is a deep question, and the essence of Loschmidt's objection to the Boltzmann equation and the $\mathcal{H}$-theorem it entails (also called *Loschmidt's paradox*) – how can irreversibility result from microscopic dynamics which are invariant under time-reversal? As Lanford (1975) stresses, "the BBGKY hierarchy is time-reversal invariant but the Boltzmann equation is not." The answer appears to be quite subtle and still somewhat contentious. Bodineau *et al.*[32] claim that the BBGKY dynamics for dilute hard spheres in d dimensions[33] converges to the solution of the Boltzmann equation, *i.e.* one must specify pre-collisional data.[34]

8.11 Appendix II: Boltzmann Equation and Collisional Invariants

Problem : The linearized Boltzmann operator $L\psi$ is a complicated functional. Suppose we replace L by $\mathcal{L}$, where

$$\mathcal{L}\psi = -\gamma\,\psi(\boldsymbol{v},t) + \gamma\left(\frac{m}{2\pi k_{\rm B}T}\right)^{3/2}\int d^3u\,\exp\left(-\frac{m\boldsymbol{u}^2}{2k_{\rm B}T}\right) \times\left\{1+\frac{m}{k_{\rm B}T}\,\boldsymbol{u}\cdot\boldsymbol{v}+\frac{2}{3}\left(\frac{m\boldsymbol{u}^2}{2k_{\rm B}T}-\frac{3}{2}\right)\left(\frac{m\boldsymbol{v}^2}{2k_{\rm B}T}-\frac{3}{2}\right)\right\}\psi(\boldsymbol{u},t) \quad . \tag{8.281}$$

Show that $\mathcal{L}$ shares all the important properties of L. What is the meaning of γ? Expand $\psi(\boldsymbol{v},t)$ in spherical harmonics and Sonine polynomials,

$$\psi(\boldsymbol{v},t) = \sum_{r\ell m} a_{r\ell m}(t)\,S^r_{\ell+\frac{1}{2}}(x)\,x^{\ell/2}\,Y_{\ell,m}(\hat{\boldsymbol{n}}), \tag{8.282}$$

with $x = mv^2/2k_{\rm B}T$, and thus express the action of the linearized Boltzmann operator algebraically on the expansion coefficients $a_{r\ell m}(t)$.

The Sonine polynomials $S^n_\alpha(x)$ are a complete, orthogonal set which are convenient to use in the calculation of transport coefficients. They are defined as

$$S^n_\alpha(x) = \sum_{m=0}^{n}\frac{\Gamma(\alpha+n+1)\,(-x)^m}{\Gamma(\alpha+m+1)\,(n-m)!\,m!} \quad , \tag{8.283}$$

and satisfy the generalized orthogonality relation

$$\int_0^\infty dx\,e^{-x}\,x^\alpha\,S^n_\alpha(x)\,S^{n'}_\alpha(x) = \frac{\Gamma(\alpha+n+1)}{n!}\,\delta_{n,n'} \quad . \tag{8.284}$$

[32] See T. Bodineau, I. Gallagher, L. Saint-Raymond, and S. Simonella, "One-sided convergence in the Boltzmann-Grad limit," *Annales de la Faculté des Sciences de Toulouse*, vol. XXVII, no. 5, p. 985 (2018).

[33] The Boltzmann-Grad limit is defined by N hard spheres each of diameter δ in a box of volume V, with $n\delta^{d-1}\ell \sim 1$, where $n = N/V \to \infty$ is the number density of spheres and ℓ is the finite mean free path. Thus $\delta \sim n^{-1/(d-1)} \to 0$.

[34] For suitable initially chaotic ensembles, Lanford's theorem yields forward Boltzmann evolution on a short kinetic-time interval. The time asymmetry is encoded in the assumptions on the initial correlations, particularly on configurations leading to collisions, and reversing the microscopic state does not preserve those assumptions.

Solution : The 'important properties' of L are that it annihilates the five collisional invariants, *i.e.* 1, $\boldsymbol{v}$, and v^2, and that all other eigenvalues are *negative*. That this is true for $\mathcal{L}$ can be verified by an explicit calculation.

Plugging the conveniently parameterized form of $\psi(\boldsymbol{v},t)$ into $\mathcal{L}$, we have

$$\begin{aligned}\mathcal{L}\psi = -\gamma \sum_{r\ell m} a_{r\ell m}(t)\, S^r_{\ell+\frac{1}{2}}(x)\, x^{\ell/2}\, Y_{\ell,m}(\hat{\boldsymbol{n}}) \;+\; \frac{\gamma}{2\pi^{3/2}} \sum_{r\ell m} a_{r\ell m}(t) \int\limits_0^\infty dx_1\, x_1^{1/2}\, e^{-x_1} \\ \times \int d\hat{\boldsymbol{n}}_1 \Big[1 + 2\, x^{1/2} x_1^{1/2}\, \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}_1 + \tfrac{2}{3}\big(x-\tfrac{3}{2}\big)\big(x_1-\tfrac{3}{2}\big)\Big]\, S^r_{\ell+\frac{1}{2}}(x_1)\, x_1^{\ell/2}\, Y_{\ell,m}(\hat{\boldsymbol{n}}_1) \quad ,\end{aligned} \tag{8.285}$$

where we've used

$$u = \sqrt{\frac{2k_{\rm B}T}{m}}\, x_1^{1/2} \qquad , \qquad du = \sqrt{\frac{k_{\rm B}T}{2m}}\, x_1^{-1/2}\, dx_1 \quad . \tag{8.286}$$

Now recall $Y_{0,0}(\hat{\boldsymbol{n}}) = \frac{1}{\sqrt{4\pi}}$ and

$$\begin{aligned} &Y_{1,1}(\hat{\boldsymbol{n}}) = -\sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{i\varphi} \qquad && Y_{1,0}(\hat{\boldsymbol{n}}) = \sqrt{\frac{3}{4\pi}}\, \cos\theta \qquad && Y_{1,-1}(\hat{\boldsymbol{n}}) = +\sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{-i\varphi} \\ &S^0_{1/2}(x) = 1 && S^0_{3/2}(x) = 1 && S^1_{1/2}(x) = \tfrac{3}{2} - x \quad ,\end{aligned} \tag{8.287}$$

which allows us to write

$$\begin{aligned} 1 &= 4\pi\, Y_{0,0}(\hat{\boldsymbol{n}})\, Y^*_{0,0}(\hat{\boldsymbol{n}}_1) \\ \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}_1 &= \frac{4\pi}{3}\Big[\, Y_{1,0}(\hat{\boldsymbol{n}})\, Y^*_{1,0}(\hat{\boldsymbol{n}}_1) + Y_{1,1}(\hat{\boldsymbol{n}})\, Y^*_{1,1}(\hat{\boldsymbol{n}}_1) + Y_{1,-1}(\hat{\boldsymbol{n}})\, Y^*_{1,-1}(\hat{\boldsymbol{n}}_1)\,\Big] \quad .\end{aligned} \tag{8.288}$$

We can do the integrals by appealing to the orthogonality relations for the spherical harmonics and Sonine polynomials:

$$\begin{aligned} \int d\hat{\boldsymbol{n}}\, Y_{\ell,m}(\hat{\boldsymbol{n}})\, Y^*_{\ell',m'}(\hat{\boldsymbol{n}}) &= \delta_{\ell,\ell'}\, \delta_{m,m'} \\ \int\limits_0^\infty dx\, e^{-x}\, x^\alpha\, S^n_\alpha(x)\, S^{n'}_\alpha(x) &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\, \delta_{n,n'} \quad .\end{aligned} \tag{8.289}$$

Integrating first over the direction vector $\hat{\boldsymbol{n}}_1$,

$$\begin{aligned}\mathcal{L}\psi = -\gamma \sum_{r\ell m} a_{r\ell m}(t)\, S^r_{\ell+\frac{1}{2}}(x)\, x^{\ell/2}\, Y_{\ell,m}(\hat{\boldsymbol{n}}) \\ + \frac{2\gamma}{\sqrt{\pi}} \sum_{r\ell m} a_{r\ell m}(t) \int_0^\infty dx_1\, x_1^{1/2}\, e^{-x_1} \int d\hat{\boldsymbol{n}}_1 \Big[Y_{0,0}(\hat{\boldsymbol{n}})\, Y^*_{0,0}(\hat{\boldsymbol{n}}_1)\, S^0_{1/2}(x)\, S^0_{1/2}(x_1) \\ + \tfrac{2}{3}\, x^{1/2} x_1^{1/2} \sum_{m'=-1}^{1} Y_{1,m'}(\hat{\boldsymbol{n}})\, Y^*_{1,m'}(\hat{\boldsymbol{n}}_1)\, S^0_{3/2}(x)\, S^0_{3/2}(x_1) \\ + \tfrac{2}{3}\, Y_{0,0}(\hat{\boldsymbol{n}})\, Y^*_{0,0}(\hat{\boldsymbol{n}}_1)\, S^1_{1/2}(x)\, S^1_{1/2}(x_1) \Big]\, S^r_{\ell+\frac{1}{2}}(x_1)\, x_1^{\ell/2}\, Y_{\ell,m}(\hat{\boldsymbol{n}}_1) \quad ,\end{aligned} \tag{8.290}$$

we obtain the intermediate result

$$\begin{aligned}\mathcal{L}\psi = -\gamma \sum_{r\ell m} a_{r\ell m}(t)\, S^r_{\ell+\frac{1}{2}}(x)\, x^{\ell/2}\, Y_{\ell,m}(\hat{\boldsymbol{n}}) \\ + \frac{2\gamma}{\sqrt{\pi}} \sum_{r\ell m} a_{r\ell m}(t) \int_0^\infty dx_1\, x_1^{1/2}\, e^{-x_1} \Big[Y_{0,0}(\hat{\boldsymbol{n}})\, \delta_{l,0}\, \delta_{m,0}\, S^0_{1/2}(x)\, S^0_{1/2}(x_1) \\ + \tfrac{2}{3}\, x^{1/2} x_1^{1/2} \sum_{m'=-1}^{1} Y_{1,m'}(\hat{\boldsymbol{n}})\, \delta_{l,1}\, \delta_{m,m'}\, S^0_{3/2}(x)\, S^0_{3/2}(x_1) \\ + \tfrac{2}{3}\, Y_{0,0}(\hat{\boldsymbol{n}})\, \delta_{l,0}\, \delta_{m,0}\, S^1_{1/2}(x)\, S^1_{1/2}(x_1) \Big]\, S^r_{\ell+\frac{1}{2}}(x_1)\, x_1^{\ell/2}.\end{aligned} \tag{8.291}$$

Appealing now to the orthogonality of the Sonine polynomials, and recalling that

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \qquad , \qquad \Gamma(1) = 1 \qquad , \qquad \Gamma(z+1) = z\,\Gamma(z) \quad , \tag{8.292}$$

we integrate over x_1. For the first term inside brackets, we invoke the orthogonality relation with $n = 0$ and $\alpha = \frac{1}{2}$, giving $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$. For the second bracketed term, we have $n = 0$ but $\alpha = \frac{3}{2}$, and we obtain $\Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2})$, while the third bracketed term leads to $n = 1$ and $\alpha = \frac{1}{2}$, also yielding $\Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2})$. Thus, we obtain the simple and pleasing result

$$\mathcal{L}\psi = -\gamma {\sum_{r\ell m}}' a_{r\ell m}(t)\, S^r_{\ell+\frac{1}{2}}(x)\, x^{\ell/2}\, Y_{\ell,m}(\hat{\boldsymbol{n}}) \tag{8.293}$$

where the prime on the sum indicates that the set

$$\mathrm{CI} = \Big\{ (0,0,0) \quad , \quad (1,0,0) \quad , \quad (0,1,1) \quad , \quad (0,1,0) \quad , \quad (0,1,-1) \Big\} \tag{8.294}$$

are to be excluded from the sum. But these are just the functions which correspond to the five collisional invariants! Thus, we learn that

$$\psi_{r\ell m}(\boldsymbol{v}) = \mathcal{N}_{r\ell m}\, S^r_{\ell+\frac{1}{2}}(x)\, x^{\ell/2}\, Y_{\ell,m}(\hat{\boldsymbol{n}}), \tag{8.295}$$

is an eigenfunction of $\mathcal{L}$ with eigenvalue $-\gamma$ if (r,ℓ,m) does not correspond to one of the five collisional invariants. In the latter case, the eigenvalue is zero. Thus, the algebraic action of $\mathcal{L}$ on the coefficients $a_{r\ell m}$ is

$$(\mathcal{L}a)_{r\ell m} = \begin{cases} -\gamma\, a_{r\ell m} & \text{if } (r,\ell,m) \notin \text{CI} \\ = 0 & \text{if } (r,\ell,m) \in \text{CI} \end{cases} \tag{8.296}$$

The quantity $\tau = \gamma^{-1}$ is the relaxation time.

It is pretty obvious that $\mathcal{L}$ is self-adjoint, since

$$\begin{aligned} \langle\, \phi \,|\, \mathcal{L}\psi \,\rangle = \langle\, \mathcal{L}\phi \,|\, \psi \,\rangle &\equiv \int\! d^3v\, f^0(\boldsymbol{v})\, \phi^*(\boldsymbol{v})\, \mathcal{L}[\psi(\boldsymbol{v})] \\ &= -\gamma n \left(\frac{m}{2\pi k_{\rm B}T}\right)^{3/2} \int\! d^3v\, \exp\left(-\frac{m\boldsymbol{v}^2}{2k_{\rm B}T}\right) \phi^*(\boldsymbol{v})\, \psi(\boldsymbol{v}) \\ &\quad + \gamma n \left(\frac{m}{2\pi k_{\rm B}T}\right)^{3} \int\! d^3v \int\! d^3u\, \exp\left(-\frac{m\boldsymbol{u}^2}{2k_{\rm B}T}\right) \exp\left(-\frac{m\boldsymbol{v}^2}{2k_{\rm B}T}\right) \\ &\qquad \times \left[1 + \frac{m}{k_{\rm B}T}\, \boldsymbol{u}\cdot\boldsymbol{v} + \frac{2}{3}\left(\frac{m\boldsymbol{u}^2}{2k_{\rm B}T} - \frac{3}{2}\right)\left(\frac{m\boldsymbol{v}^2}{2k_{\rm B}T} - \frac{3}{2}\right)\right] \phi^*(\boldsymbol{v})\, \psi(\boldsymbol{u}) \quad , \end{aligned} \tag{8.297}$$

where n is the bulk number density and $f^0(\boldsymbol{v})$ is the Maxwellian velocity distribution.